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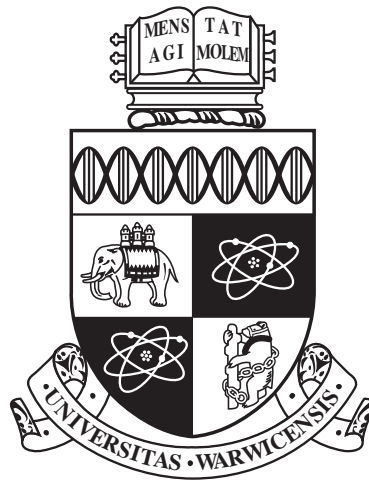
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**Self-Similar Markov Processes and the Time
Inversion Property**

by

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Declarations

This thesis is submitted to the University of Warwick in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Chapter 1 of this thesis is an introduction, with no original content except for Section 1.5. I declare that the work in Chapters 2, 3 and 4 is my own, except when specifically stated otherwise.

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Abstract

The objective of this thesis is to further the understanding of the time inversion property for self-similar Markov processes. In particular, we focus upon seeking a full characterisation of the class of processes that enjoy the time inversion property.

The first chapter in this thesis is a review of current literature in the areas that we use in the sequel. Chapter 2 provides a full characterisation of processes enjoying the time inversion property on \mathbb{R} up to certain restrictions. Namely, we show that on \mathbb{R}^+ , the only processes that enjoy the time inversion property are Bessel processes in the wide sense. Extending this characterisation to \mathbb{R} , we show that we are necessarily restricted to variations of Bessel and Dunkl processes. We then give an expression of the semigroup density that all processes with the time inversion property must satisfy. In Chapter 3, we extend some of these results to \mathbb{R}^n . We provide a restriction on the jump measure of processes with the time inversion property and show that $\hat{\rho}(R_t)$ is necessarily a Bessel process for a process R_t with the time inversion property and a defined function $\hat{\rho}$. Finally, Chapter 4 extends the work of Vuolle-Apiala [2012] on the skew product representation and presents a methodology by which one can construct examples of processes with the time inversion property. This leads to several examples of particular interest.

Chapter 1

Introduction

In the theory of applied stochastic processes, we are often looking for some property seen empirically in experiment that can be used to provide tractability in our models. For example, the unpredictability of stock prices suggests that the next increment of a stock price process cannot be predicted with any degree of accuracy. This leads to the Markov property used in modelling: future states of a Markov process depend only upon the present state and not upon past states. Moreover, most experimental data is taken at discrete intervals, but it is often modelled as a continuous process that is sampled at time intervals. It is felt that this is more realistic in light of what is actually happening. By looking even more closely, we can see that these increments often appear independent of each other. Indeed, much of the interest in Markov and Lévy processes can be accounted for by the observations of these simple assumptions in the “natural” world.

Once we have rationalised independent increments, we are then often searching for some distribution that can be seen in the experimental data or that follows logically from a combination of our current understanding and outputs that we would like from the model. Returning to our example of stock prices, this can be seen in our assumptions. Considering the vast size of the stock market and the vast number of agents acting on a particular price process, it is not a far stretch to assume that there are a significant number of people affecting the price of any particular stock, each with independent views on what the market will do next. Thus, the overall sum of a large number of independent random agents acting on the process suggests that we should use a stable distribution; most famously, the normal distribution.

A similar thing is seen when the large number of independent collisions by air molecules on a smoke molecule are viewed under a microscope. As it would be impossible to include every single collision by a molecule in any model, we describe the movement of the smoke molecule probabilistically. This leads to the modelling of this phenomenon as the Brownian motion of the particle. Considering this as a subject of a near infinite number of independent collisions also leads to a scaling property for Brownian motion B_t given by

$$((B_t : t \geq 0), \mathbb{P}_0) \stackrel{(d)}{=} ((cB_{c^{-2}t} : t \geq 0), \mathbb{P}_0)$$

for all $c > 0$ where $\stackrel{(d)}{=}$ means equality in distribution.

By the same token, a scaling property similar to this can be seen in large number of experiments, see Embrechts and Maejima [2002]. That is, there is a relationship between the distribution of a process on a large scale over a long period of time and the distribution of that same process on a smaller scale after a much shorter period of time. This has led to the self-similarity condition. Rigorously, this is defined within stochastic models as a process X_t that has the following equivalence in law

$$((X_t : t \geq 0), \mathbb{P}_x) \stackrel{(d)}{=} \left((cX_{\frac{t}{c^H}} : t \geq 0), \mathbb{P}_{\frac{x}{c}} \right) \quad (1.1)$$

where c is a positive constant, x is the point from which we initiate the process and $H \in \mathbb{R}$.

Since the paper by Lamperti [1972], who introduced a set of strong Markov positive processes with this condition, much has been done in this area, see [Pardo and Rivero, 2013] for a review. Lamperti was also able to express these processes as time-changed exponential Lévy processes. This bijective correspondence is at the heart of the vast majority of work that has been conducted in the area of positive self-similar Markov processes. Mainly because Lévy processes have been extensively studied for decades; their stationary and independent increments allow them to be easily determined and permit many useful constructions.

In the case when $H = 2$ in (1.1), which we call the 2-self-similar case, this scaling property leads to an interesting question about the process' distribution in time. What happens if you replace the constant c in the distribution by the time variable? Is $tX_{\frac{1}{t}}$ still a homogeneous Markov process?

Processes that fulfil this property are said to enjoy the time inversion property and it is this class of processes that we predominantly consider in this thesis.

The time inversion property was first identified in Brownian motion by Lévy. Later, Shiga and Watanabe [1973] and Watanabe [1975] broadened the scope of this characteristic by examining a variant of the time inversion property. For a diffusion with continuous paths X_t , Watanabe considered the properties of a process $Y_t = g_t(X_{\frac{1}{t}})$, where $\{g_t(\cdot)\}_{t>0}$ described a family of homeomorphisms. This reduces to the case considered in this thesis when $g_t(x) = tx$. Watanabe then proved that any Bessel process in the wide sense retained all the properties of a diffusion under this inversion when $g_t(x) = tx$. Thus, all Bessel processes in the wide sense have the time inversion property.

Extending the idea of time inversion, Gallardo and Yor [2005] proved a sufficient condition pertaining to the semigroup density of a process that guaranteed the time inversion property under certain constraints. This cemented time inversion under the Watanabe functional $g_t(x) = t^\alpha x$ for some α . The expression of the semigroup density enabled the exploration of several new and previously unknown examples of processes with the time inversion property, such as the Dunkl processes (see Gallardo and Yor [2006]) and their extensions, through the explicit expressions of their semigroups. Furthermore, Gallardo and Yor were also able to provide examples of several instances when the law of the time inverted process could be written as an h -transform of the original process. This, in itself, led to several interesting properties concern-

ing these processes. Taking this a step further, Lawi [2008] provided a necessary and sufficient condition by which one could determine whether a process had the time inversion property, provided that the original process had a semigroup density that was absolutely continuous with respect to the Lebesgue measure. It is this condition that we shall primarily use in the sequel.

It is now known, see Watanabe [1975] and Lawi [2008], that Bessel processes in the wide sense are the only examples of diffusions on \mathbb{R}^+ enjoying the time inversion property. Furthermore, as far as we are aware, the only known examples of processes enjoying the time inversion property on \mathbb{R} are Brownian motion and generalised Dunkl processes, see Chybiryakov et al. [2008]; Gallardo and Yor [2006]; Lawi [2008]; Rösler and Voit [1998].

The aim of this thesis is to extend the characterisation of the time inversion property for Markov processes on \mathbb{R}^n . We also construct several new examples of this class of processes and consider several properties associated with them.

Organisation and Outline of this Thesis

In the remainder of this introduction, we give a short outline of some of the concepts that we use in this thesis. Firstly, we give a brief survey of the stochastic processes that we utilise in the later chapters together with some of their basic properties. This outline will largely focus on self-similar Markov processes (ssMps). In this review, we first consider the self-similarity property for the general class of ssMps on \mathbb{R}^n , before considering abstractions of these processes restricted to \mathbb{R}^+ and \mathbb{R} given by the Lamperti and Lamperti-Kiu representations respectively. These expressions prove invaluable in gaining an intuitive characterisation of these processes. Section 1.2 gives an outline of the time inversion property itself as described in Gallardo and Yor [2005] and this will form the bedrock of the thesis. Sections 1.3 and 1.4 consider two examples of classes of self-similar processes enjoying this property; namely, the Bessel processes and the Dunkl processes. In Section 1.5, we provide a specialised example of a self-similar Markov process, which does not enjoy the time inversion property, that corroborates with a statement made in Lawi [2008]. Section 1.6, provides a brief review of the skew product representation. Finally, in Section 1.7 we describe an expression of the infinitesimal generator operator that lends itself to the self-similar property.

Chapter 2 investigates the time inversion property on \mathbb{R} . Initially focussing on \mathbb{R}^+ , we show that under a restriction on the absolute continuity of the semigroup, the only processes enjoying this property are the Bessel processes. This result then leads to a full characterisation of processes on \mathbb{R} enjoying this property under the same restriction. To do this, we first show that the radial part of any process with the time inversion property, when appropriately scaled, is a Bessel process. This leads to a restriction on the behaviour of this class away from zero that can be used to find a representation for any possible excursions from the origin.

In Chapter 3, we aim to extend this characterisation to \mathbb{R}^n . We start by explicitly determine the possible jumps that this class of processes can make before considering the continuous paths of the processes and a mapping between these processes and the Bessel processes. This allows us to express a characterisation through the generator of the process.

Finally, inspired by Vuolle-Apiala [2012], in Chapter 4 we investigate the skew product representation of processes in \mathbb{R}^n and its connection with the time inversion property. This representation allows us to construct numerous new examples of processes enjoying the time inversion property that fall beyond the scope of Gallardo and Yor [2005] and Lawi [2008].

1.1 Outline of Self-Similar Markov Processes

In recent times there has been a greater demand for stochastic models with the self-similar property, for example, see Leland et al. [1994]; López-Ardao et al. [2000]. Empirical observations of this property in experiment have led to a large amount of literature on Markov processes that satisfy the abstract and rigorous definition of self-similarity given by (1.1), see Embrechts and Maejima [2002].

In this section, we present a brief outline of self-similar Markov processes focussing on their restrictions to \mathbb{R}^+ and \mathbb{R} before discussing an important relationship between these processes and Lévy processes. Namely, the Lamperti representations on \mathbb{R}^+ and \mathbb{R} given in Lamperti [1972] and Chaumont et al. [2013] respectively. Although we give a more general overview here, which uses the case $H > 0$ in (1.1), the rest of this thesis will predominantly focus on the case $H = 2$. It is always possible to find a map from the case $H = 2$ to $H > 0$ if required.

1.1.1 Basic Properties of pssMps

The term positive self-similar Markov process (or pssMp) was first coined by Lamperti in a paper (Lamperti [1972]) in which he was able to show the Feller and killing properties of these processes. In fact, Lamperti took pssMp to mean positive semi-stable Markov process, but the definition has since evolved to self-similar, which is equivalent, but better reflects the intended meaning. Following this work, we define a positive self-similar Markov process R in the following way.

Definition 1. *If \mathbb{P}_x is the law of a process $R := (R_t)_{t \geq 0}$ initiated at $x \in [0, \infty)$, then R is a pssMp if it is a strong Markov process taking values on $[0, \infty)$ with the equivalence in law given by (1.1) for any $c, x > 0$, where $H > 0$ is a fixed constant called the index of self-similarity of the process R .*

Furthermore, Lamperti [1972] was also able to determine all unkilld processes of this form through their generator. Applied to a function $f \in C_0^2(\mathbb{R}^+)$, the generator of a pssMp R is necessarily of the form

$$\mathcal{A}_R f(x) = \mu x^{1-H} f'(x) + \frac{\sigma^2}{2} x^{2-H} f''(x) + \int_0^\infty (f(y) - f(x) - l_x(y) x f'(x)) n(x, dy)$$

where $\mu, \sigma > 0$; $l_x(y)$ is a function such that $l_x(y) \sim y$ inside a small neighbourhood of x and vanishes outside this neighbourhood and $n(x, dy)$ is a Lévy kernel that satisfies the condition $c^H n(cx, dc) = n(x, dy)$.

Unfortunately, in the case of continuous paths, this generator is limiting. The only diffusions that fall into the category of pssMps are time-scaled Bessel processes and their powers, where the self-similarity follows from the self-similarity of Brownian motion and their construction as a generalisation of the Euclidean norm in varying dimensions.

In the case of processes with jumps, the examples become more interesting. All stable subordinators fall into the category of pssMps; the self-similar property following from the stability of the Lévy process. This category of processes also contains stable processes under certain h -transforms, for example Doney [2005] uses the renewal measure as a transform to condition the process to stay positive whilst retaining self-similarity.

Despite the usefulness and real world applications of the self-similar and positivity properties, arguably the most important property of pssMps is the isomorphism between the class of pssMps and the class of Lévy processes.

1.1.2 The Lamperti Transform

In this section, we outline an important bijection between a - possibly killed - Lévy process ξ sent to a cemetery state $\Delta = \{-\infty\}$ at time e_q and a pssMp R . Here e_q is an independent exponential distribution that represents the killing time of the process. This bijection will involve both a spacial functional change and an additive time-change in order to move from the stationary increments of a Lévy process to the strictly non-negative and scaling properties of a pssMp.

Theorem 2 (Lamperti [1972]). *$R := (R_t)_{0 \leq t \leq T_0}$ is a pssMp with index of self-similarity H initiated at $x > 0$ and stopped at T_0 , its first hitting time of zero, if and only if there exists a Lévy process $\xi := (\xi_t)_{t \geq 0}$ such that*

$$R_t^{(x)} = x \exp \left(\xi_{H(x^{-H}t)} \right).$$

In this case, the time change A_t is given by

$$A_t = \int_0^t e^{H\xi_s} ds$$

and H_t is its right inverse. Furthermore, R is absorbed at zero at time $x^H A_\infty$ that corresponds with the lifetime of the Lévy process that is possibly killed and sent to the cemetery $\Delta = \{-\infty\}$ at an exponential time e_q .

Amongst a vast number of other properties, Caballero and Chaumont [2006b] utilise this to induce a trichotomy in pssMps. Each pssMp falls into one of the following three exhaustive and distinct categories with probability one depending on the characteristics of the Lévy process that generates it. We use the notation $\zeta^{(x)}$ to denote the lifetime of a pssMp initiated at x , that is, the (possibly infinite) time taken by the process to reach its cemetery state at zero.

1. In the first case, $\zeta^{(x)} = \infty$ for all $x > 0$ and the pssMp will never hit zero and will

eventually drift to infinity almost surely. This corresponds to a process whose underlying Lévy process is an unkilld process that satisfies the property $\limsup_{t \rightarrow \infty} \xi_t = \infty$.

2. In the second case, $\zeta^{(x)} < \infty$ for all $x > 0$ and the pssMp hits zero continuously, that is, $R_{\zeta^{(x)}-} = 0$. This happens when the pssMp is induced by an unkilld Lévy process with the property $\lim_{t \rightarrow \infty} \xi_t = -\infty$.
3. Finally, it is possible that $\zeta^{(x)} < \infty$ for $x > 0$, but the pssMp jumps downwards to zero. This corresponds to a pssMp generated by a Lévy process that is killed and sent to the cemetery state $\Delta = \{-\infty\}$ at an exponentially distributed time.

If e_q is an exponentially distributed random variable, then the time change $H(x^H \zeta^{(x)})$ is also exponentially distributed. This is necessary to preserve the Markov property and was shown by Lamperti [1972] and is exposed in [Kyprianou, 2014, p. 375 Lemma 13.3].

1.1.3 Extending a pssMp killed at zero

Given the two latter cases in the trichotomy, it seems natural to ask if we can provide a conservative pssMp that is also recurrent. One way of constructing such a process is through a recurrent extension, see Rivero [2005], and involves a “re-issuing” of the process from zero at an exponentially distributed time after it has been killed.

In order to do this, we first determine an entrance law by taking the weak limit of the measure of the process as its starting point tends to zero. It can be shown through duality, see Bertoin and Yor [2002], that this weak limit exists provided the underlying Lévy process, associated by the Lamperti transform, has finite expectation in finite time.

Now that we have this entrance law, we can construct the recurrent extension of the process. Therefore, given a pssMp \tilde{R} that is initiated at $x > 0$ and absorbed at the origin at time $\zeta^{(x)}$, we construct a recurrent extension R such that $R \stackrel{(d)}{=} \tilde{R}$ up until time $\zeta^{(x)}$. Then we construct an excursion measure that determines the excursions of the process away from zero. In this way, we can produce processes that may not only leave zero continuously, but may also jump from zero after an exponentially distributed waiting time to coincide with the law and Markov property of the original process.

In the case of pssMps with continuous paths almost surely, it has been shown in Rivero [2007] that the recurrent extension is unique.

1.1.4 Self-Similar Markov Processes on \mathbb{R} and the Lamperti-Kiu Representation

Markov processes satisfying property (1.1) on \mathbb{R} and stopped at their first hitting time of zero satisfy an extension of this Lamperti representation. Known as the Lamperti-Kiu representation, it takes into account a switching between positive and negative self-similar processes at exponentially distributed jumping times. This representation is discussed more fully in Chaumont et al. [2013], but we give a brief summary here.

Let R be a self-similar Markov processes on \mathbb{R} killed at its first hitting time of zero. For Lévy processes ξ^+, ξ^- ; real-valued random variables U^+, U^- and exponentially distributed random variables ζ^+, ζ^- with parameters $q^+, q^- > 0$ respectively, we define a sequence of random variables

$$(\xi^{(x,k)}, U^{(x,k)}, \zeta^{(x,k)}) = \begin{cases} (\xi^{(+,k)}, U^{(+,k)}, \zeta^{(+,k)}) & \text{if } \text{sgn}(x)(-1)^k = 1 \\ (\xi^{(+,k)}, U^{(-,k)}, \zeta^{(-,k)}) & \text{if } \text{sgn}(x)(-1)^k = -1 \end{cases}$$

where each $\xi^{(+,k)}, U^{(+,k)}, \zeta^{(+,k)}$ (respectively $\xi^{(-,k)}, U^{(-,k)}, \zeta^{(-,k)}$) defines an independent set of random variables with the distribution ξ^+, U^+, ζ^+ (ξ^-, U^-, ζ^- respectively). Additionally, we denote the sum of the exponentially distributed times and the resulting counting process by

$$T_n^{(x)} = \sum_{k=0}^{n-1} \zeta^{(x,k)} \quad \text{and} \quad N_t^{(x)} = \max\{n \geq 0 : T_n^{(x)} \leq t\}.$$

With this notation, we define the Feller multiplicative process Y as

$$Y_t^{(x)} = x \exp \left(\xi_{t - T_{N_t^{(x)}}^{(x)}} + \sum_{k=0}^{N_t^{(x)} - 1} \left(\xi_{\zeta^{(x,k)}}^{(x,k)} + U^{(x,k)} \right) + i\pi N_t^{(x)} \right). \quad (1.2)$$

Intuitively, this Feller multiplicative process started at $x > 0$ is equivalent in distribution to a time-changed exponential Lévy process ($x \exp(\xi_t^+)$) up until its first change of sign or hitting of zero, whichever comes sooner. At the time of its first change of sign, which occurs at an exponentially distributed time, the process jumps from its current position to the negative of this position multiplied by the exponential of a real-valued random variable ($\exp(U^+)$). The process then behaves similarly on the negative half-line using the distributions ξ^-, U^- and ζ^- .

For a 2-self-similar process, the Lamperti-Kiu representation is then given by the stochastic process and time-change

$$R_t = Y_{H_t^{(x)}}^{(x)} \quad \text{and} \quad H_t^{(x)} = \int_0^t Y_s^{(x)2} ds.$$

Using the intuitive explanation of Y_t , it can be seen that the process is equivalent in distribution to a killed pssMp up until its first change of sign.

For more information, we refer the reader to Chaumont et al. [2013].

In the sequel, we use the notation $LK(\xi^+, \xi^-, U^+, U^-, q^+, q^-)$ to denote a process R with the Lamperti-Kiu representation given above.

1.2 Time Inversion

The time inversion property, as the name suggests, stems from invariance of processes under the inversion of their time variable. There are two major interpretations of the time inversion property although one is a particular case of the other.

The first, and earliest, interpretation is time inversion in the sense of Watanabe [1975]. Restricting himself to conservative diffusions on the positive reals, Watanabe defined the time inversion property for a set $Q \subseteq \mathbb{R}^+$ and a Radon measure μ as follows.

Definition 3 (Time Inversion in sense of Watanabe [1975]). *A conservative diffusion process $X := (X_t)_{t>0}$ with initial measure μ is said to have the time inversion property on a set $Q \subseteq \mathbb{R}^+$ if there exists a second conservative diffusion $X' := (X'_t)_{t>0}$ with initial measure μ' on a set $Q' \subseteq \mathbb{R}^+$ and a time dependent function $g(t, x) : [0, \infty) \times Q \rightarrow Q'$, which is continuous for all $t > 0$, such that*

$$\left(g(t, X_{\frac{1}{t}}), t \geq 0\right) \stackrel{(d)}{=} (X'_t, t \geq 0).$$

It can be shown, see Watanabe [1975], that on \mathbb{R}^+ it is necessary for the function g to take the form $g(t, x) = tx$ and this has led to a second interpretation of the time inversion property extended to all Markov processes.

The second interpretation of the time inversion property is time inversion in the sense of Gallardo and Yor [2005]. In this case, if we take $X := ((X_t)_{t>0}, \mathbb{P}_x)$ to be any time homogeneous Markov process on \mathbb{R}^n , for any $n \in \mathbb{N}$, initiated at an $x \in \mathbb{R}^n$, then, in general, its inverted process $t^\alpha X_{\frac{1}{t}}$ is a time inhomogeneous Markov process for $\alpha > 0$.

If we fix an $x \in \mathbb{R}^n$ and define the inverted process for an α by $Y_t^{(x)} := t^\alpha X_{\frac{1}{t}}$, then, provided that the semigroup density of X is absolutely continuous with respect to the Lebesgue measure and given by $p_t(x, y)$, Gallardo, Yor and Lawi were able to express the distribution of the time inverted process as follows. For any Borel f and $0 < s < t$

$$\mathbb{E}_x \left[f(t^\alpha X_{\frac{1}{t}}) | s^\alpha X_{\frac{1}{s}} = a \right] = \int f(y) q_{s,t}^{(x)}(a, y) dy, \quad (1.3)$$

where

$$q_{s,t}^{(x)}(a, y) = \frac{1}{t^n} \frac{p_{\frac{1}{t}}(x, \frac{y}{t}) p_{\frac{1}{s}-\frac{1}{t}}(\frac{y}{t}, \frac{a}{s})}{p_{\frac{1}{s}}(x, \frac{a}{s})}. \quad (1.4)$$

Gallardo and Yor [2005] then say a process enjoys the time inversion property if it is a time homogeneous Markov process $(X_t)_{t>0}$ on \mathbb{R}^n whose inverted process $(t^\alpha X_{\frac{1}{t}})_{t>0}$ is also a time homogeneous Markov process on \mathbb{R}^n for all starting points $x \in \mathbb{R}^n$. It is this definition, see also Definition 4 below, of the time inversion property (in the sense of Gallardo and Yor [2005]) that we refer to in the sequel.

Gallardo and Yor also noted that (1.4) implies that any Doob h -transform of X given by

$$\mathbb{P}_x^\phi |_{\mathcal{F}_t} = \frac{\phi(X_t)}{\phi(x)} e^{-\lambda t} |_{\mathcal{F}_t} \quad (1.5)$$

for a function ϕ and $\lambda \in \mathbb{R}$ leads to the same inverted process with the same semigroup density

$q^{(x)}$. Other space-time functions $g(x, t) \neq \phi(x)e^{\lambda t}$ were omitted as they do not lead to time homogeneous Markov processes. Thus, under these conditions, the class of processes satisfying the time inversion property is defined up to an h -transform of the form (1.5), see Doob [1957] and Revuz and Yor [2005]. On account of this property the remainder of this thesis aims only to classify all processes enjoying the time inversion property up to a conservative Doob h -transform.

We also take this opportunity to state that from now on, when we say the time inversion property and do not explicitly give a degree, we mean the time inversion property of degree $\alpha = 1$.

With the time inversion property established, a natural question is to ask which processes have the time inversion property. To determine this, Lawi [2008] first makes the following assumptions on our time homogeneous Markov process X with state space $S \in \{\mathbb{R}^+, \mathbb{R}, \mathbb{R}^n\}$. For a fixed $t \geq 0$:

(H1) The semigroup density of X is absolutely continuous with respect to the Lebesgue measure and therefore, the probability kernel can be written

$$P_t(x, dy) = p_t(x, y)dy;$$

(H2') The semigroup density $p_t(x, y)$ is twice differentiable in x, y and t for all $(t, x, y) \in (0, \infty) \times \mathring{S} \times \mathring{S}$, where \mathring{S} is the interior of S .

In the case where (H1) and (H2') are satisfied, Gallardo, Yor and Lawi were able to use the semigroup density to classify processes that enjoyed the time inversion hypothesis of Pitman and Yor [1981]

$$i(Q_x^{(y)}) = Q_y^{(x)},$$

where $(Q_x^{(y)})_{x \in \mathbb{R}^n}$ is the family of laws induced by the semigroup density $q^{(x)}$ and i is the mapping that takes a probability measure \mathbb{P} of a right-continuous process X_t to that of $t^\alpha X_{\frac{1}{t}-}$. Namely, if X is a time homogeneous Markov process that satisfies (H1) and (H2'), then it enjoys the time inversion property of degree α if and only if its semigroup density takes the form

$$p_t(x, y) = \frac{1}{t^{\frac{n}{2}}} \Phi\left(\frac{x}{t^{\frac{\alpha}{2}}}, \frac{y}{t^{\frac{\alpha}{2}}}\right) \theta\left(\frac{y}{t^{\frac{\alpha}{2}}}\right) \exp\left(-\rho\left(\frac{x}{t^{\frac{\alpha}{2}}}\right) - \rho\left(\frac{y}{t^{\frac{\alpha}{2}}}\right)\right) \quad (1.6)$$

or if X is in h -transform with a semigroup of this form. Furthermore, Gallardo, Yor and Lawi also deduced the following restrictions on the functions Φ , θ and ρ . For any $\lambda > 0$ and any $x, y, z \in S$

$$\begin{aligned} \Phi(\lambda x, y) &= \Phi(x, \lambda y) \\ \theta(\lambda z) &= \lambda^\beta \theta(z) \\ \rho(\lambda z) &= \lambda^2 \rho(z) \end{aligned}$$

where $\beta \in \mathbb{R}$ is fixed. This is a result we shall use extensively in the sequel.

This provides a somewhat tractable restriction on the functions θ and ρ , in their radial part at least. However, it does not provide such a useful restriction on Φ . For this reason, under the restriction that Φ is symmetric, Gallardo and Yor [2005] also deduce the reproducing identity in terms of θ and ρ

$$\Phi(x, a) = \int \Phi(x, y) \Phi(y, a) \theta(y) \exp(-(\rho(a) + \rho(x) + \rho(y))) dy.$$

Furthermore, we also note that any h -transform of a process with a semigroup density of this form will also enjoy the time inversion property, see Lawi [2008].

This permits us to list several examples of processes with the time inversion property: Bessel processes in the wide sense (Pitman [1975], [Revuz and Yor, 2005, Chapter XI.1]); generalised Dunkl processes (Gallardo and Yor [2006], Rösler and Voit [1998], Demni) and Wishart processes (Bru [1991]). We discuss the first two of these in the next two sections.

It was also shown in Gallardo and Yor [2005] that, when (H1) and (H2') are satisfied, the generator of the inverted process $(Y_t^{(x)})_{t>0}$ can be expressed in terms of the generator of the original process R and the ‘opérateur carré du champ’ given in [Revuz and Yor, 2005, Chapter VIII.3]. If a function f is in the domain of the generator of the original process \mathcal{A}_R , then the generator of the inverted process $\mathcal{A}_Y^{(x)}$ can be written

$$\mathcal{A}_Y^{(x)} f(b) = \mathcal{A}_R f(b) + \frac{1}{\Phi(x, b)} \Gamma(\Phi(x, \cdot), f)(b),$$

where $\Gamma(f, g) = \mathcal{A}_R(fg) - f\mathcal{A}_R g - g\mathcal{A}_R f$ for f in the domain of the generator is the ‘opérateur carré du champ’, see [Revuz and Yor, 2005, Chapter VIII.3]. This means that the choice of x determines what Gallardo and Yor refer to as the Bessel drift or Dunkl drift in the case of Bessel and Dunkl processes respectively. Similarly, the family of laws $\{Q_b^{(x)}\}_{b \in \mathbb{R}^n}$ of the inverted process $Y^{(x)}$ has the following h -transform relationship

$$Q_b^{(x)}|_{\mathcal{F}_t} = \frac{\Phi(x, R_t)}{\Phi(x, b)} e^{-t\rho(x)} \cdot P_b|_{\mathcal{F}_t},$$

where $\{P_b\}_{b \in \mathbb{R}^n}$ is the family of laws of the original process. This, in turn, also leads to an expression for the semigroup density of the inverted process

$$q_t^{(x)}(b, y) = \frac{\Phi(x, y)}{\Phi(x, b)} e^{-t\rho(x)} p_t(b, y).$$

In this thesis, we consider the time inversion property in a slightly more general setting of a cone. Defining a cone as a non-trivial set $S \subseteq \mathbb{R}^n$ such that $x \in S$ implies that $ax \in S$ for all $a > 0$, we define the time inversion property.

Definition 4. A time homogeneous Markov process $(X_t)_{t>0}$ taking values on a cone $S \subseteq \mathbb{R}^n$ and initiated at x is said to have the time inversion property if, for all $x \in \dot{S}$, its inverted process

$(tX_{\frac{1}{t}})_{t>0}$ is a time homogeneous Markov process on S .

The main aim of this thesis is to classify the set of all processes that have this property. In Chapter 2, we consider all processes with the time inversion property on \mathbb{R} and classify processes enjoying the time inversion property on each cone subset of this space, \mathbb{R}^+ , \mathbb{R}^- and \mathbb{R} . To do this, we take the slightly weaker version of (H2')

(H2) The semigroup density $p_t(x, y)$ is twice differentiable in x, y and t for all $(t, x, y) \in (0, \infty) \times S \setminus \{0\} \times S \setminus \{0\}$

and a conservative assumption on the process. Chapters 3 and 4 extend these results to all of \mathbb{R}^n using the additional assumption on the function ρ

(H4) The function ρ in (2.2) is continuous and positive for all $x \in \mathbb{R}^n \setminus \{0\}$ vanishing only at the origin.

1.3 The Bessel Process in the Wide Sense

The squared Bessel processes of dimension $\delta \in \mathbb{N}$, $\delta > 0$, arise as the square of the Euclidean norm of a δ -dimensional Brownian motion on \mathbb{R}^+ . That is, for a δ -dimensional Brownian motion $\underline{B}_t = (B_t^{(1)}, \dots, B_t^{(\delta)})^T$ initiated at a point $z \in \mathbb{R}^\delta \setminus \{0\}$, we set the squared Bessel process of dimension δ to be $Q_t^{(\delta)} := \|\underline{B}_t\|^2 = (B_t^{(1)})^2 + \dots + (B_t^{(\delta)})^2$. From this and the smoothness properties of the Euclidean norm, we can determine the infinitesimal generator of the process applied to a function $f \in C^2(\mathbb{R}^+)$ to be

$$\mathcal{A}_Q f(x) = \delta f'(x) + 2x f''(x)$$

for an $x > 0$. This also permits a generalisation to the full class of squared Bessel process for any dimension $\delta > 0$.

Furthermore, by the additive dimensional properties of the probability measure bestowed upon Q by the squared Euclidean norm it is possible to determine the distribution of the squared Bessel process through the Laplace transform. This is exposed in [Revuz and Yor, 2005, Chapter XI Corollary XI.1.4],

$$\mathbb{E}_x \left[e^{\lambda Q_t^{(\delta)}} \right] = \frac{\exp \left(-\frac{\lambda x}{1+2\lambda t} \right)}{(1+2\lambda t)^{\frac{\delta}{2}}},$$

from this we can explicitly obtain the semigroup density of the process through a Laplace inversion

$$q_t^\delta(x, y) = \frac{1}{2} I_\nu \left(\frac{\sqrt{xy}}{t} \right) \left(\frac{y}{x} \right)^{\frac{\nu}{2}} \exp \left(-\frac{x+y}{2t} \right),$$

where ν is the index of the process given by $\nu = \frac{\delta}{2} - 1$. We can employ this to calculate the semigroup density of the Bessel process itself. Defining $r_t^{(\nu)} := \sqrt{Q_t^{(\nu)}}$, we can compute the

semigroup density as

$$q_t^\delta(x, y) = I_\nu\left(\frac{xy}{t}\right) \left(\frac{y}{x}\right)^\nu \frac{y}{t} \exp\left(-\frac{x^2 + y^2}{2t}\right)$$

and on account of the fact that the square root function is bijective on \mathbb{R}^+ , the new process retains the Markov property. Using the smoothness of the square root function on \mathbb{R}^+ , its generator applied to a function $f \in C^2(\mathbb{R}^+)$ is given by

$$\mathcal{A}_R f(x) = \frac{2\nu + 1}{2x} f'(x) + \frac{1}{2} f''(x)$$

for any $x > 0$ and $\nu > -1$. For more details on the Bessel process, we refer the reader to [Revuz and Yor, 2005, Chapter XI.1], Pitman and Yor [1981] and Göing-Jaeschke and Yor [2003].

1.3.1 The Wide Sense

The class of Bessel processes in the wide sense generalises the class of Bessel processes given above to also include all their conservative h -transforms. That is, we would like to include all the processes with a generator applied to a function $f \in C^2(\mathbb{R}^+)$ of the form

$$\mathcal{A}f(x) = \left(\frac{2\nu + 1}{2x} + \frac{h'(x)}{h(x)}\right) f'(x) + \frac{1}{2} f''(x),$$

where ν is the index of the Bessel process, for a function h that preserves the conservative property. In order to do this, h must be a normalised solution to the eigenvalue problem with the infinitesimal generator of the Bessel process as the differential operator. Thus, h , which solves the equation $\mathcal{A}h(x) - ch(x) = 0$, for an index ν and $c > 0$ is given, up to a positive multiplicative constant, by

$$h_c^\nu(x) = \frac{2^\nu \Gamma(\nu + 1) I_\nu(x\sqrt{2c})}{(x\sqrt{2c})^\nu}.$$

Equivalently, for integer dimensions δ (where $\delta = 2\nu + 2$), it has been shown by Rajabpour [2009] that this set of processes is equivalent to the Euclidean norm of a Brownian motion plus drift as opposed to just a Brownian motion in the case of a Bessel process.

In this thesis we note that we take the class of Bessel processes in the wide sense to include both the Bessel processes themselves and all of their conservative h -transforms.

1.4 The Dunkl Process

The self-similar property is one that is viewed often in experiment. It results from a natural scale between time and displacement in a variety of physical, chemical and economic problems. For this reason, there has often been much motivation for the construction of malleable self-similar processes that can be extended beyond the positive reals. The Dunkl martingale is one

proposed solution to this obstacle.

The Dunkl martingale is the result of perturbing the differential operator associated with the infinitesimal generator of Brownian motion in a way that preserves the scaling property. That is, if we take the following perturbed differential operator T_k applied to a function $f \in C^1(\mathbb{R})$

$$T_k f(x) = f'(x) + k \left(\frac{f(x) - f(-x)}{x} \right)$$

and then use this as the differential operator in the generator for Brownian motion, we obtain the Dunkl operator. When this is applied to a function $f \in C^2(\mathbb{R} \setminus \{0\})$, we have the infinitesimal generator of the Dunkl martingale defined on $\mathbb{R} \setminus \{0\}$ as

$$\mathcal{A}_k f(x) = f''(x) + k \left(\frac{2}{x} f'(x) - \frac{f(x) - f(-x)}{x^2} \right)$$

where $k \geq 0$ measures the perturbation of the differential operator and gives a parameter of the Dunkl martingale. This terminology is justified by the fact that the process is a martingale, see Gallardo and Yor [2006]. It can also be seen that the self-similar property passed down from the unperturbed generator of Brownian motion still holds, that is,

$$\mathcal{A}_k f(c \cdot) \left(\frac{x}{c} \right) = c^2 \mathcal{A}_k f(x)$$

for all $c > 0$.

If we now consider a Feller process whose semigroup is generated by this differential operator, we can immediately see that we have a martingale process whose absolute value will be a Bessel process with continuous paths. In fact, an extension of Gilat's Theorem, see [Gallardo and Yor, 2006, Theorem 1], states that this process represents the unique martingale whose absolute value is a Bessel process with index $\nu = k - \frac{1}{2}$. Another curious conclusion that can be drawn from this or from the expression of the generator is that the Lévy kernel associated with the Dunkl process only permits jumps from positive to negative whilst maintaining the same absolute value. This gives an intuitive perception of the one-dimensional Dunkl process as a Bessel process alternating between positive and negative values.

The Dunkl martingales can be generalised further by relaxing the martingale condition. Once the martingale condition is relaxed, we can split the jump and drift coefficients of the process, that is, we can write the extended Dunkl generator of a generalised Dunkl process applied to a function $f \in C^2(\mathbb{R})$ as

$$\mathcal{A}_{(\nu, \lambda)} f(x) = f''(x) + \frac{2\nu + 1}{2x} f'(x) - \lambda \frac{f(x) - f(-x)}{x^2}$$

for $\nu > -1$, and $\lambda \geq 0$. This provides us with an intuitive understanding of a Dunkl process as a 2-self-similar process whose absolute value is a Bessel process of index ν and that jumps to points of opposing sign with rate $\frac{\lambda}{x^2}$.

This intuition can be expressed more rigorously by defining the Dunkl process as $D_t = r_t(-1)^{N_{H_t}^{(\lambda)}}$ up to the first hitting time of zero, see Gallardo and Yor [2006]. Here, $N_t^{(\lambda)}$ is the Poisson process with rate λ which is time changed by the additive process $H_t = \int_0^t \frac{1}{|D_s|^2} ds = \int_0^t \frac{1}{r_s^2} ds$, where r is the Bessel process resulting as the absolute value of D . Thus, the semigroup density can be computed explicitly as the sum of a positive and negative Bessel process that alternate with the parity of a Poisson process, which is time-changed by an additive process whose distribution is equivalent to the Hartman-Watson distribution, see Hartman and Watson [1974]. This computation is done in Gallardo and Yor [2006] and gives a reasonably simple expression of the semigroup density

$$\begin{aligned} \mathbb{P}_x(D_t \in dy) &= D_{(\nu, \lambda)}\left(\frac{xy}{t}\right) |y|^{2\nu+1} \frac{1}{t^{\nu+1}} e^{-\frac{x^2+y^2}{t}} dy \\ &= \frac{1}{2} \mathbf{1}_{[y \geq 0]} \left[I_\nu\left(\frac{xy}{t}\right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} + I_{\sqrt{\nu^2+4\lambda}}\left(\frac{xy}{t}\right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} \right] \\ &\quad + \frac{1}{2} \mathbf{1}_{[y < 0]} \left[I_\nu\left(\frac{xy}{t}\right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} - I_{\sqrt{\nu^2+4\lambda}}\left(\frac{xy}{t}\right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} \right], \end{aligned}$$

where $D_{(\nu, \lambda)}\left(\frac{xy}{t}\right)$ is the extended Dunkl kernel, see Appendix A.

This alternation seen in Gallardo and Yor [2006] also leads to a pleasant skew product representation of the Dunkl process in the case where D starts from $x \neq 0$ and $\nu > 0$ as

$$D_t = Y_{H_t}, \tag{1.7}$$

$$Y_u = \exp\left(B_u + \nu u + i\pi N_u^{(\lambda)}\right), \tag{1.8}$$

where B_u is a Brownian motion independent of the Poisson process $N_u^{(\lambda)}$. This is the first of many associations between the time inversion property and a skew product representation that dominate the subject of Chapter 4.

Furthermore, an extension of the absolute continuity relationship between Bessel processes also applies to the Dunkl process and if \mathbb{P}^k is the probability measure of a Dunkl martingale with parameter k , then for $\nu > 0$, or equivalently $k, l > \frac{1}{2}$, we can construct the Radon-Nikodym derivative between the measures of two Dunkl martingales as

$$\left. \frac{d\mathbb{P}_x^l}{d\mathbb{P}_x^k} \right|_{\mathcal{F}_t} = \left(\frac{|D_t|}{|x|} \right)^{l-k} \left(\frac{l}{k} \right)^{N_t} \exp\left(-\frac{l^2 - k^2}{2} H_t \right),$$

where N_t is the number of sign changes of D_t within the interval $(0, t)$.

1.4.1 The Dunkl Process in More Than One Dimension

As the Dunkl process extends the Bessel process to the negative reals, it is natural to ask whether there is a process that extends the Bessel process to several dimensions. This process relies on a similar construction to the above and in this construction it retains many of the original properties. We first extend the perturbed Dunkl operator $(T_i, (i = 1, \dots, n))$ to \mathbb{R}^n for

a reflection group W through the operator applied to a function $f \in C^1(\mathbb{R}^n)$ given by

$$T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \cdot \frac{f(x) - f(\sigma_\alpha(x))}{\alpha \cdot x},$$

where R is a root system and R_+ is its positive part; σ_y is a reflection in the hyperplane orthogonal to y given by

$$\sigma_y(x) = x - 2 \frac{y \cdot x}{\|y\|^2} y$$

and k is a multiplicity function acting on the root system R . That is, it is invariant under the action of the associated reflection group W . It can be seen that this reduces to the earlier case when $n = 1$.

Using this perturbed Dunkl differential operator in the same way as before gives the Dunkl Laplacian

$$\mathcal{A}_k f(x) = \frac{1}{2} \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \left[\frac{\nabla f(x) \cdot \alpha}{\alpha \cdot x} + \frac{f(\sigma_\alpha(x)) - f(x)}{(\alpha \cdot x)^2} \right]$$

for a function $f \in C_0^2(\mathbb{R}^n)$. It can once again be seen that this generator satisfies the self-similar property. Furthermore, this process can also be extended by relaxing the martingale condition and allowing for separate jump and drift coefficients in much the same way as the one-dimensional case. This reveals what is often referred to as the generalised Dunkl process on \mathbb{R}^n

$$\mathcal{A}_{k,\lambda} f(x) = \frac{1}{2} \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla f(x) \cdot \alpha}{\alpha \cdot x} + \sum_{\alpha \in R_+} \lambda(\alpha) \frac{f(\sigma_\alpha(x)) - f(x)}{(\alpha \cdot x)^2}$$

where λ is a non-negative multiplicity function.

Once again, this implies that the generalised Dunkl process is only permitted to jump from a point to its image under one of the reflection operators σ_α associated with the root system R . Thus, it can only jump between Weyl chambers on \mathbb{R}^n . Considering this, we can construct a Dunkl process as a radial part restricted to a single Weyl chamber and determined by the following generator applied to $f \in C^2(\mathbb{R}^n)$

$$\mathcal{A}_k^W f(x) = \frac{1}{2} \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\nabla f(x) \cdot \alpha}{\alpha \cdot x}$$

that then jumps between Weyl chambers according to the reflection operator σ_α . Thus, it can be written as a composition depending on the number of roots in the root system. For $R = \{\alpha_1, \dots, \alpha_l\}$ then if Y_t^0 is the radial part given above we can construct the process through induction as

$$Y_t^i = \sigma_{\alpha_i}^{N_{A_t}^i} Y_t^{i-1} \quad i = 1, \dots, l$$

where $A_t^i = \int_0^t \frac{1}{(\alpha \cdot Y_s^{i-1})^2} ds$, see Chybiryakov [2008].

1.5 An Example of a Positive 2-Self-Similar Markov Process that does not Satisfy the Time Inversion Property

In Lawi [2008, p. 3, Corollary 2.5] the following statement was made:

Corollary 5. *Lawi [2008] A Markov process that enjoys the time inversion property of degree α is a self-similar process of index $\frac{2}{\alpha}$, or is in h -transform relationship with it. The converse is not true*

However, no further explanation was given regarding the converse so, in the interests of clarity, we provide an example satisfying Lawi's assumptions that acts as an advocate in this corollary. Although we give this initially in the case of the time inversion property of degree 1, the example can be easily extended to any degree of time inversion.

As our proposed example of a 2-self-similar process we look at the process $Z_t = T_t^{\frac{1}{4}}$. That is, the fourth root of the $\frac{1}{2}$ -stable subordinator process, which represents the first passage times of a one-dimensional Brownian motion. From Rogers and Williams [2000], we know that T has a semigroup density that is absolutely continuous with respect to the Lebesgue measure and is given by

$$\mathbb{P}_x(T_t \in dy) = \frac{t}{\sqrt{2\pi(y-x)^3}} \exp\left(-\frac{t^2}{2(y-x)}\right) dy.$$

First and foremost, the Markov property of the fourth root of this process follows from the bijective property of the function $f(x) = x^{\frac{1}{4}}$ on \mathbb{R}^+ . If we now consider the self-similar property we can see that the process T_a is self-similar with index $H = \frac{1}{2}$ from the semigroup density. For a $c > 0$,

$$\begin{aligned} \mathbb{P}_{cx}\left(\frac{1}{c}T_{\sqrt{ct}} \in dy\right) &= \frac{\sqrt{ct}}{\sqrt{2\pi c^3(y-x)^3}} \exp\left(-\frac{ct^2}{2c(y-x)}\right) cdy \\ &= \frac{t}{\sqrt{2\pi(y-x)^3}} \exp\left(-\frac{t^2}{2(y-x)}\right) dy \\ &= \mathbb{P}_x(T_t \in dy). \end{aligned}$$

Furthermore, through this property, we can show that Z is 2-self-similar

$$\mathbb{P}_{cz}\left(\frac{1}{c}Z_{c^2t} \in dy\right) = \mathbb{P}_{cz^4}\left(\frac{1}{c}(T_{c^2t})^{\frac{1}{4}} \in dy\right) = \mathbb{P}_{z^4}\left(\frac{1}{c}(c^4T_t)^{\frac{1}{4}} \in dy\right) = \mathbb{P}_z(Z_t \in dy)$$

and therefore, Z is a pssMp. To denounce the time inversion property, we can compute the new

semigroup density. For any Borel f on \mathbb{R}^+

$$\begin{aligned}\mathbb{E}_z[f(Z_t)] &= \mathbb{E}_{z^4}\left[f\left(T_t^{\frac{1}{4}}\right)\right] = \int f\left(y^{\frac{1}{4}}\right) \frac{t}{\sqrt{2\pi(y-x)^3}} \exp\left(-\frac{t^2}{2(y-x)}\right) dy \\ &= \int f(y) \frac{4y^3 t}{\sqrt{2\pi(y^4-x^4)^3}} \exp\left(-\frac{t^2}{2(y^4-x^4)}\right) dy.\end{aligned}$$

Thus, the semigroup density of our proposed 2-self-similar process is given by

$$\mathbb{P}_x(Z_t \in dy) = \frac{4y^3 t}{\sqrt{2\pi(y^4-x^4)^3}} \exp\left(-\frac{t^2}{2(y^4-x^4)}\right) dy.$$

This does not have the form given in (2.2), or even (2.6) on \mathbb{R}^+ , despite satisfying (H1) and (H2'), and therefore it does not have the time inversion property

Remark 6. *This process can be used as a counter-example to the equivalence of self-similarity and the time inversion property of any degree. For example, this can be done for the time inversion property of degree α for $\alpha > 0$ and self-similar property of index $H = \frac{2}{\alpha}$ simply by taking the process $Z_t = T_t^{\frac{\alpha}{4}}$.*

1.6 The Skew Product

In the previous section, we saw that the Dunkl process could be represented as a skew product representation (1.7). This representation was composed of an independent Poisson process and Brownian motion with drift both time-changed by the additive process H_t resulting from the right inverse of the time change $A_t = \int_0^t e^{2B_u} du$. Using the Lamperti transform on this time-changed Brownian motion, we have the basis of the skew product representation of the Dunkl process,

$$D_t = r_t \exp\left(i\pi N_{H_t}^{(\lambda)}\right),$$

where $r := (r_t)_{t \geq 0}$ is a Bessel process with index $\nu \geq 0$.

The skew product representation itself on the complex plane, when it exists, is a representation of a stochastic process where the radial part is given by a Bessel process and the angular part follows some time-changed independent stochastic process. That is, in the two-dimensional case the process can be expressed on the complex plane

$$R_t := r_t e^{i\gamma_{H_t}}$$

where r is a Bessel process of index $\nu \geq 0$ and $\gamma := (\gamma_t)_{t \geq 0}$ is an independent Markov process time-changed by

$$H_t = \int_0^t \frac{1}{r_u^2} du.$$

We note that self-similarity is a consequence of the Bessel nature of the radial part.

Other than the Dunkl process notable examples of the skew product representation include

1. Planar Brownian motion: $Z_t = X_t + iY_t$ for independent Brownian motions X and Y on the complex plane can be expressed as $r_t e^{iW_{H_t}}$, where r is a Bessel process of index zero and W is another independent Brownian motion, see Yor [2001b].
2. The Dunkl process in \mathbb{R}^n for $n \geq 2$ provided the process avoids the origin almost surely, see Chybiryakov [2008].
3. Rotation invariant self-similar diffusions, see Graversen and Vuolle-Apiala [1986a] and Vuolle-Apiala [1986].

In the n -dimensional case the skew product representation extends to

$$(r_t, \Theta_{H_t})_{r_0 > 0, t \geq 0} \quad (1.9)$$

where the Bessel process r denotes the radial part and Θ is the angular part of the process, which is itself an independent stochastic process.

1.7 The Expression for the Infinitesimal Generator of a Positive Self-Similar Markov Process

The form of the infinitesimal generator that we shall use throughout this thesis is taken from [Kolokoltsov, 2011, Chapter 5]. In Kolokoltsov [2011], it is shown that a generator given by the following when applied to a function $f \in \mathcal{D}_{\mathcal{A}}$:

$$\mathcal{A}f(x) = \sum_{i=1}^n \mu_i(x) \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^n \sigma_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \int (f(x+y) - f(x) - l_\epsilon(y) \cdot \nabla f(x)) a(x) \nu(dy)$$

generates a Feller semigroup where the core of the domain is $C_\infty^2(\mathbb{R}^n)$ provided that the measure ν satisfies $\int (1 \wedge \|y\|^2) \nu(dy) < \infty$ and the matrix $\Sigma := \{\sigma_{ij}(x)\}_{i,j=1}^n$ is a matrix of functions that is always positive definite. Here, μ is a vector of functions and $l_\epsilon(x)$ is a function such that $l_\epsilon(x) \sim x$ close to the origin and that vanishes outside $B_\epsilon(0)$.

In order to make things slightly easier, we make the simple substitution $y = x + y$ inside the integral, which leaves us with

$$\mathcal{A}f(x) = \sum_{i=1}^n \mu_i(x) \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^n \sigma_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \int (f(y) - f(x) - l_\epsilon(y-x) \cdot \nabla f(x)) n(x, dy), \quad (1.10)$$

where $n(x, dy) = a(x)\nu(dy - x)$. This slight alteration means that the Lévy kernel of a process $X := (X_t)_{t \geq 0}$ on \mathbb{R}^n with semigroup generated by this generator now satisfies the compensation formula from Bass [1979] and Jacod and Shiryaev [1987]

$$\mathbb{E}_x \left[\sum_{s \leq t} f(X_s, X_{s-}) \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^n \setminus \{R_s\}} f(y, R_s) n(R_s, dy) ds \right]$$

and is more easily associated with the extended Itô formula. For these reasons, when we consider a 2-self-similar process in the sequel we use the form of the generator given by

$$\mathcal{A}f(x) = \sum_{i=1}^n \mu_i(x) \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^n \sigma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \int (f(y) - f(x) - l_\epsilon(y - x) \cdot \nabla f(x)) n(x, dy), \quad (1.11)$$

where, in contrast to (1.10), σ_{ij} is constant and $c^2 n(cx, dc y) = n(x, dy)$ for all $c > 0$. The restrictions follow from the self-similarity of the generator itself.

We also note that in much of the literature, when restricted to the case of pssMps on \mathbb{R}^+ , the form of the generator given by Lamperti [1972]

$$\mathcal{A}f(x) = \mu x^{1-\alpha} f'(x) + \frac{\sigma^2}{2} x^{2-\alpha} + \int \left(f(ux) - f(x) - \tilde{l}_\epsilon(\log u) x f'(x) \right) x^\alpha \Pi(d \log u)$$

is used. However, if we make the substitution $xu = y$ in the integral

$$\mathcal{A}f(x) = \mu x^{1-\alpha} f'(x) + \frac{\sigma^2}{2} x^{2-\alpha} + \int \left(f(y) - f(x) - \tilde{l}_\epsilon(\log y - \log x) x f'(x) \right) x^\alpha \Pi(d \log y - \log x)$$

we obtain a representation which is equivalent to our representation restricted to \mathbb{R}^+ with $\tilde{l}_\epsilon(\log y - \log x) x = l_\epsilon(y - x)$ and $x^\alpha \Pi(d \log y - \log x) = n(x, dy)$ or equivalently, $\Pi(d \log y) = n(1, dy)$ with the self-similarity restriction.

Chapter 2

A Characterisation of the Time Inversion Property on \mathbb{R}

2.1 Introduction

In this chapter, we consider a property closely related to the self-similar property for Feller processes. Let X be a stochastic process. If inverting the time variable of the process and multiplying by that same time variable to the power of some index, $t^\alpha X_{\frac{1}{t}}$, preserves the Markov and time homogeneous properties, then we say the process has the time inversion property of degree α . For more information, we refer the reader to Section 1.2.

The aim of this chapter is to obtain a full characterisation of all Markov processes on \mathbb{R} enjoying the time inversion property (Definition 4), subject to their semigroup densities being absolutely continuous with respect to the Lebesgue measure. Initially, we take the case of just the positive (or negative) reals, given by \mathbb{R}^+ (or \mathbb{R}^-), as a state space. We take both these spaces to include the point $\{0\}$. Given a few restrictions on the semigroup density, we employ the results of Lawi [2008] to show that the Bessel processes in the wide sense are the only processes on \mathbb{R}^+ that enjoy the time inversion property. Taking this further, we then consider processes on the entire real line stopped at their first hitting time of zero and prove that, up to an h -transform, any process with the time inversion property can be uniquely determined through its Lamperti-Kiu representation. This representation leads to a necessary and sufficient form of the semigroup density that guarantees the time inversion property. Finally, we look at recurrent extensions of these processes and determine a necessary and sufficient condition for any Markov process to enjoy the time inversion property on \mathbb{R} .

The structure of this chapter is as follows. In Section 2, we discuss some preliminaries and notation that will be needed in the sequel. This includes a discussion of the main assumptions used in the paper together with an overview of both the time inversion property itself and the notation we use to denote the Lamperti-Kiu representation on \mathbb{R} . Section 3 considers the time inversion property, given in Definition 4, when the process is restricted to \mathbb{R}^+ and shows that the only processes satisfying this property are Bessel processes in the wide sense. Finally, in Section 4 we extend our scope to consider any processes on \mathbb{R} . On this domain we give an

explicit characterisation of all processes enjoying the time inversion property on \mathbb{R} , subject to our assumptions. Initially we provide a characterisation for processes up to their first hitting time of zero. This characterisation then permits us to explicitly describe the semigroup densities of any process belonging to this class by considering all possible excursions away from the origin.

2.2 Preliminaries and Notation

In this section, we state some assumptions that we use throughout this chapter. We also consider some preliminaries related to the time inversion property, which we shall use extensively in the sequel.

For the remainder of this chapter, let $\{(R_t : t \geq 0), \mathbb{P}_x\}$ be a Markov process on a state space $S \in \{\mathbb{R}^+, \mathbb{R}^-, \mathbb{R}\}$ upon which we make the following assumptions for any fixed $t > 0$:

(H1) The semigroup density of R is absolutely continuous with respect to the Lebesgue measure and therefore, the probability kernel can be written

$$P_t(x, dy) = p_t(x, y)dy;$$

(H2') The semigroup density $p_t(x, y)$ is twice differentiable in x, y and t for all $(t, x, y) \in (0, \infty) \times \mathring{S} \times \mathring{S}$;

(H3) The process R is conservative over its state space S , that is to say

$$1 = \int_S p_t(x, y)dy$$

for all $x \in \mathring{S}$.

We remark that the above assumptions are sufficient to invoke the results of Lawi [2008] on \mathbb{R} , although we later exchange (H2') for a slightly weaker assumption, see Remark 7. It is also true that some authors include the conservative property in the definition of Markov semigroup. Here we have only stated that R is a Markov process and therefore, we take the conservative property to be a separate assumption (H3).

Following the work of Gallardo and Yor [2005], $(t^\alpha R_{\frac{1}{t}} : t > 0)$ is a Markov process, which is time inhomogeneous in general. As mentioned in Section 1.2, Gallardo and Yor [2005] were able to define the time inverted process up to an h -transform through the process $Y_t^{(x)} := t^\alpha R_{\frac{1}{t}}$ for a fixed $x \in S$. The distribution of this process can be expressed as (1.3). In the case where the assumptions (H1), (H2') and (H3) are satisfied, this also allows us to define the semigroup density. If $p_t(x, y)$ is the semigroup density of the time homogeneous Markov process R , then the semigroup density of the time inverted process of degree 1 is given up to an h -transform by

the distribution of $Y^{(x)}$,

$$p_{s,t}^{Y^{(x)}}(a,b) = \frac{1}{t^n} \frac{p_{\frac{1}{t}}(x, \frac{b}{t}) p_{\frac{1}{s}-\frac{1}{t}}(\frac{b}{t}, \frac{a}{s})}{p_{\frac{1}{s}}(x, \frac{a}{s})}.$$

We now have that R has the time inversion property of degree α if $Y^{(x)}$ is a time homogeneous Markov process. This is equivalent to being able to write the distribution as

$$\mathbb{E}_a \left[f(Y_{t-s}^{(x)}) \right] = \mathbb{E}_x \left[f(t^\alpha R_{\frac{1}{t}}) | s^\alpha R_{\frac{1}{s}} = a \right] \quad (2.1)$$

for $t \geq s$. Furthermore, if R satisfies (H1) and (H2'), then Gallardo and Yor [2005] and Lawi [2008] have shown that R enjoys the time inversion property of degree α on a state space of dimension n if and only if the semigroup is of the form

$$p_t(x, y) = \frac{1}{t^{\frac{n}{2}}} \Phi \left(\frac{x}{t^{\frac{\alpha}{2}}}, \frac{y}{t^{\frac{\alpha}{2}}} \right) \theta \left(\frac{y}{t^{\frac{\alpha}{2}}} \right) \exp \left(-\rho \left(\frac{x}{t^{\frac{\alpha}{2}}} \right) - \rho \left(\frac{y}{t^{\frac{\alpha}{2}}} \right) \right) \quad (2.2)$$

or if R is in h -transform with a semigroup of this form for restrictions on the functions Φ , θ and ρ given by

$$\Phi(\lambda x, y) = \Phi(x, \lambda y) \quad (2.3)$$

$$\theta(\lambda z) = \lambda^\beta \theta(z) \quad (2.4)$$

$$\rho(\lambda z) = \lambda^2 \rho(z) \quad (2.5)$$

for any $\lambda > 0$ and a fixed $\beta \in \mathbb{R}$. In the following, we shall refer to the index on (2.4) as $\beta = 2\nu + 1$, upon which we will later prove the restriction $\nu > -1$. The additional $\nu > -1$ restriction will be proved on \mathbb{R}^+ in Theorem 10 and on \mathbb{R} in Theorem 12. Lawi's result on the form of the semigroup density (2.2) is a result we shall use extensively in the sequel.

Remark 7. *When considering the time inversion property on $S = \mathbb{R}$, we replace assumption (H2') with the slightly weaker assumption:*

(H2) *The semigroup density $p_t(x, y)$ is twice differentiable in x, y and t for all $(t, x, y) \in (0, \infty) \times S \setminus \{0\} \times S \setminus \{0\}$.*

This is still sufficient to invoke the results of Lawi [2008] on the restricted domain $\mathbb{R} \setminus \{0\}$. We can see this by following the same proof for the necessity condition as Lawi [2008] on this restricted domain and using the simple extension of the Euler Homogeneous Function Theorem, given in [Apostol [1981] page 364-365], for functions that are not differentiable at zero. The semigroup density will therefore have the same form as (2.2) on the domain $\mathbb{R} \setminus \{0\}$. We characterise the time inversion property by initially considering processes that either do not hit

zero almost surely or are stopped at zero so the semigroup density on this restricted domain is sufficient.

To extend the results to processes that continue after hitting zero, we consider all possible recurrent extensions when using the stopped processes as a minimal process and therefore do not require the semigroup density result to apply at the origin.

For the remainder of this chapter, we use (H1-3) to denote the assumptions (H1), (H2) and (H3).

2.3 The Time Inversion Property on \mathbb{R}^+

It has been shown previously that the only diffusions on \mathbb{R}^+ that enjoy the time inversion property are Bessel processes in the wide sense, see Watanabe [1975]. Our main result on \mathbb{R}^+ extends this statement from diffusions to a much wider class of Markov processes. Namely, the class of all Markov processes satisfying assumptions (H1-3).

We prove this result by using the results of Lawi [2008] concerning the semigroup density of the process. However, it will first be necessary to reduce the form of the semigroup density from the more general case on \mathbb{R}^n to a more restricted version, which is relevant to only \mathbb{R}^+ .

Lemma 8. *Let R be a Markov process on \mathbb{R}^+ with a semigroup density of the form (2.2) satisfying (H1-3).*

- (i) *The functions θ and ρ in (2.2) satisfy $\theta(z), \rho(z) > 0$ for all $z \in \mathbb{R}^+ \setminus \{0\}$.*
- (ii) *The semigroup density of the process R must be of the more restricted form*

$$p_t(x, y) = \phi\left(\frac{xy}{t}\right) \frac{y^{2\nu+1}}{t^{\nu+1}} \exp\left(-\frac{1}{2\sigma^2 t}(x^2 + y^2)\right). \quad (2.6)$$

Proof. (i) We first deal with the positivity restriction for the functions ρ and θ , which we prove by contradiction.

If we first assume that there exists a z such that $\theta(z) = 0$, then, by (2.4), $\theta(z) = 0$ for all $z \in \mathbb{R}^+ \setminus \{0\}$ and therefore the semigroup density is null, $p_t(x, z) = 0$, for all $z > 0$ by the continuity (H2). This contradicts (H3).

To prove the statement for ρ , we begin by making a similar assertion. If we assume that there exists a $z \in \mathbb{R}^+ \setminus \{0\}$ such that $\rho(z) = 0$, then by (2.5) $z^2 \rho(1) = 0$ implying that $\rho(1)$ also vanishes. Thus, using (2.5) again, $\rho(y) = y^2 \rho(1) = 0$ for all $y \in S$, that is, ρ vanishes everywhere in the state space.

This, coupled with $\theta(y) = \theta(1)|y|^{2\nu+1} = K|y|^{2\nu+1}$ by (2.4), provides us with a more simplified version of the semigroup density. Moreover, by (H3) we have that for any $x \in \mathring{S}$

$$1 = \int \Phi(1, xy) K|y|^{2\nu+1} dy,$$

where we have set $t = 1$ and used (2.3) since $x > 0$. Substituting $xy = u$ into the integral

$$x^{2\nu+2} = \int \Phi(1, u) K|u|^{2\nu+1} du$$

for any $x > 0$. Since the right hand side of the equation is constant, with no dependence on x , this is a contradiction.

(ii) Now that we have positivity, Lawi [2008] has shown that on this particular state space the functions involved in the semigroup given above in (2.2) must satisfy the following properties: $\Phi(x, y) = \Phi(1, xy) = \phi(xy)$; $\theta(y) = y^{2\nu+1}$ and $\rho(y) = \frac{y^2}{2\sigma^2}$ for $\sigma > 0$. This follows easily from the individual restrictions (2.3), (2.4) and (2.5) applied only to the positive reals. Thus, the semigroup density on \mathbb{R}^+ is given by the more simplified semigroup density given in the statement of the lemma by equation (2.6).

□

From Lemma 8, it follows that finding the possible processes with this semigroup density is equivalent to determining the required choice of the function ϕ . Furthermore, from equation (2.6) ϕ can be expressed as

$$\phi(z) = p_1(z, 1) \exp\left(\frac{1}{2\sigma^2}(z^2 + 1)\right) \quad (2.7)$$

and by assumption (H2) and the continuity of the exponential function we have that $\phi \in C^2(\mathbb{R}^+)$.

Remark 9. *Note that we have chosen the constant $\rho(1) = \frac{1}{2\sigma^2}$ rather than a simple constant. This has been chosen for two reasons. Firstly, we would like to be consistent with the work of Lawi [2008], who uses this particular constant, and secondly, this choice of constant lends itself more easily to the scaling property we shall use later.*

With these restrictions on the semigroup and the conservative property we can now fully characterise processes enjoying the time inversion property, see 4, on \mathbb{R}^+ .

Theorem 10. *If $R := (R_t)_{t>0}$ is a Markov process on \mathbb{R}^+ satisfying assumptions (H1-3), then it enjoys the time inversion property if and only if it is a, possibly time-scaled, Bessel process in the wide sense.*

This result extends in the obvious way to the negative reals, \mathbb{R}^- .

Proof. By Lemma 8, a Markov process on \mathbb{R}^+ that enjoys the time inversion property and complies with the restrictions (H1-2) above has a semigroup density of the form

$$p_t(x, y) = \phi\left(\frac{xy}{t}\right) \frac{y^{2\nu+1}}{t^{\nu+1}} \exp\left(-\frac{1}{2\sigma^2 t}(x^2 + y^2)\right)$$

for some $\sigma > 0$, or it is possibly in h -transform with this process. For the purposes of this proof we initially concern ourselves only with the processes that have a semigroup density of this

form and deal with their respective h -transforms later. By the assumption (H3), our semigroup density is conservative on \mathbb{R}^+ , so for all $x \in \mathbb{R}^+$

$$1 = \int_0^\infty \phi\left(\frac{xy}{t}\right) \frac{y^{2\nu+1}}{t^{\nu+1}} e^{-\frac{x^2+y^2}{2\sigma^2 t}} dy. \quad (2.8)$$

We would like to target the function ϕ so we make the substitution $y = \frac{2\sqrt{u}\sigma^2 t}{x}$ to simplify its argument

$$1 = \int_0^\infty \phi(2\sigma^2\sqrt{u}) u^\nu 2^\nu \sigma^{2\nu+2} \frac{2^{\nu+1} \sigma^{2\nu+2} t^{\nu+1}}{x^{2\nu+2}} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{2u\sigma^2 t}{x^2}} du.$$

We now introduce a constant that is conducive for determining the function ϕ via a Laplace transform. Letting $\mu = \frac{2\sigma^2 t}{x^2}$ yields

$$1 = \int_0^\infty \phi(2\sigma^2\sqrt{u}) u^\nu 2^\nu \sigma^{2\nu+2} \mu^{\nu+1} e^{-\frac{1}{\mu}} e^{-\mu u} du$$

or equivalently, with some simplification

$$\frac{e^{\frac{1}{\mu}}}{\mu^{\nu+1}} = \int_0^\infty \phi(2\sigma^2\sqrt{u}) u^\nu 2^\nu \sigma^{2\nu+2} e^{-\mu u} du. \quad (2.9)$$

Thus, by the injective property of the Laplace transform for continuous functions (Lerch's Theorem); the Laplace transform of a Bessel function given in [Gradshteyn and Ryzhik, 2007, p. 709 Formula 6.643.2] and the continuity of ϕ shown in (2.7)

$$\phi(2\sigma^2\sqrt{u}) u^{\frac{\nu}{2}} \sigma^{2\nu+2} 2^\nu = I_\nu(2\sqrt{u})$$

and therefore, the function ϕ can be expressed in terms of a Bessel function

$$\phi(z) = \frac{1}{\sigma^2 z^\nu} I_\nu\left(\frac{z}{\sigma^2}\right)$$

for $\nu > -1$. This restriction exists because when $\nu \leq -1$ we do not have the equality (2.9) for any ϕ and consequently, (2.8) cannot hold. This violates condition (H3).

We can now write the semigroup density using this unique representation (up to time-scaling and an index ν) of the function ϕ

$$p_t(x, y) = I_\nu\left(\frac{xy}{\sigma^2 t}\right) \frac{y^{\nu+1}}{\sigma^2 t x^\nu} \exp\left(-\frac{1}{2\sigma^2 t}(x^2 + y^2)\right).$$

Thus, a process with the time inversion property either has a semigroup density of this form or it is in h -transform with one of these processes. This implies that any process that enjoys the time inversion property and is restricted by (H1-3) is a Bessel process in the wide sense after a possible deterministic time-scaling.

The converse follows because the semigroup density of a Bessel process satisfies the

assumptions (H1-3) and has a semigroup density of the form (2.2) so it enjoys the time inversion property. As noted in Section 1.2, this has already been shown in Lawi [2008]. \square

We remark that Bessel processes with indexes $\nu < -1$ are not included in this Theorem. This is because they hit zero almost surely and have no reflection that provides a conservative process.

2.4 A Characterisation of the Time Inversion Property on \mathbb{R}

In this section, we use the previous result on \mathbb{R}^+ to fully characterise the set of processes with the time inversion property on \mathbb{R} , see Definition 4. We begin by showing that, up to an h -transform, $2\rho(R_t)$ is a squared Bessel process. At this point we then separate any potential processes into the process up until it first hits zero and its extension. Considering processes up until their first hitting time of zero, we show that processes with the time inversion property on the real line are limited to Bessel processes (which are restricted to \mathbb{R}^+ or \mathbb{R}^-), generalised Dunkl processes and their h -transforms. Finally, we investigate all possible extensions of these processes that still permit the time inversion property. From this, we can characterise any processes with the time inversion property explicitly through their semigroup density.

On \mathbb{R} we note that, by the results of Lawi [2008], any process satisfying (H1-3) that also enjoys the time inversion property must have a semigroup density of the form (2.2) or be in h -transform with a process of this form. Therefore, we focus on identifying all processes with a semigroup density of the form (2.2), whilst noting that our results apply to all processes with the time inversion property up to an h -transform.

2.4.1 Equivalence with a Squared Bessel Process

In this section we show that, up to an h -transform, $2\rho(R_.)$ is equivalent in distribution to a squared Bessel process for any process R on \mathbb{R} that has the Gallardo and Yor semigroup density given in (2.2). Thus, we note that any Markov process satisfying the time inversion property and (H1-3) is in h -transform with a process that satisfies this equivalence in distribution.

Much like the similar result on \mathbb{R}^+ given in Lemma 8, we once again reduce the form of the general semigroup density on \mathbb{R}^n given in Lawi [2008] to a form that is only applicable to \mathbb{R} . We do this by restricting the forms of the functions ρ and θ .

Lemma 11. *Let R be a process with a semigroup density of the form (2.2) on the state space \mathbb{R} (not \mathbb{R}^+) restricted by assumptions (H1-3) as above. Then the additional restrictions on the individual functions given by (2.4) and (2.5) imply that θ must be of the form*

$$\theta(z) = \begin{cases} K_1 |z|^{2\nu+1} & \text{if } z \geq 0 \\ K_2 |z|^{2\nu+1} & \text{if } z < 0 \end{cases} \quad (2.10)$$

and, similarly, up to a time-scaling of R_t , ρ must take the form

$$\rho(z) = \begin{cases} \frac{z^2}{2} & \text{if } z \geq 0 \\ \frac{z^2}{2\sigma^2} & \text{if } z < 0 \end{cases} \quad (2.11)$$

for $K_1, K_2, \sigma > 0$.

From now on, we say that a process X has a certain property up to a time-scaling if there exists a constant $c > 0$ such that X_{ct} has the required property. For example R_t with a semigroup density of the form (2.2) has a function ρ in this semigroup density of the form (2.11) if there exists a $c > 0$ such that R_{ct} has a function ρ of this form.

Proof. If we consider the function θ in the semigroup (2.2), then by (2.4) we have that $\theta(z) = |z|^{2\nu+1}\theta(1)$ for $z > 0$ and $\theta(z) = |z|^{2\nu+1}\theta(-1)$ for $z < 0$. However, unlike the previous case, it is now possible that $K_1 = \theta(1) \neq \theta(-1) = K_2$ and therefore, in full generality, the function θ takes the form

$$\theta(z) = \begin{cases} K_1 |z|^{2\nu+1} & \text{if } z \geq 0 \\ K_2 |z|^{2\nu+1} & \text{if } z < 0. \end{cases}$$

Furthermore, we must also show that $K_1, K_2 > 0$. since θ is a direct multiple in the form of the semigroup (2.2), the non-negativity of the two constants is guaranteed by the non-negativity of the semigroup density. This means we are only required to show that neither of the constants vanish. If we assume that the constant K_2 vanishes, then $\theta(z) = 0$ for all $z < 0$. In turn, this means that $p_t(x, z) = 0$ for all $t > 0, x \in \mathbb{R}$ and all $z < 0$ effectively excluding \mathbb{R}^- from the state space. This returns us to the results of the previous section regarding processes on the positive reals so we omit this case. Similarly, we omit the case when $K_1 = 0$.

Analogously, we also consider the function ρ . Once again, using the restriction (2.5), we can see that

$$\rho(z) = \begin{cases} z^2 \rho(1) & \text{if } z \geq 0 \\ z^2 \rho(-1) & \text{if } z < 0. \end{cases}$$

where we pay heed to the possibility that $\rho(1) \neq \rho(-1)$.

However, we must also guarantee the positivity of these two constants. To do so, we first assume that $\rho(1) = 0$ and thus, $\rho(z) = 0$ for all $z > 0$ by (2.5). Consider $\mathbb{P}_x(R_t > 0)$. With the above restrictions on the semigroup density coupled with (2.3), for any $t, x > 0$ we have

$$\mathbb{P}_x(R_t > 0) = \int_0^\infty \Phi\left(1, \frac{xy}{t}\right) K_1 \frac{y^{2\nu+1}}{t^{\nu+1}} dy < 1.$$

Making the substitution $u = xy$ and letting $t = \frac{s}{z}$ for $s, z > 0$ yields

$$x^{2\nu+2} z^{-\nu-1} > \int_0^\infty \Phi\left(z, \frac{u}{s}\right) K_1 \frac{u^{2\nu+1}}{s^{\nu+1}} du.$$

Since this is true for any $x > 0$, then by letting x be arbitrarily close to zero for $\nu > -1$ we have

$$0 = \int_0^\infty \Phi\left(z, \frac{u}{s}\right) K_1 \frac{u^{2\nu+1}}{s^{\nu+1}} du,$$

or equivalently $\mathbb{P}_z(R_s > 0) = 0$ for any $z, s > 0$. This effectively removes the positive half-line from our state space reducing to the case of the previous section and so we omit this particular case. The proof is similar for $\rho(-1) > 0$.

This implies that on the whole of \mathbb{R} , ρ must be of the form:

$$\rho(z) = \begin{cases} \frac{z^2}{2k_1^2} & \text{if } z \geq 0 \\ \frac{z^2}{2k_2^2} & \text{if } z < 0. \end{cases}$$

where $\rho(1) = \frac{1}{2k_1^2}$ and $\rho(-1) = \frac{1}{2k_2^2}$. Equivalently, without loss of generality we can consider a time-scaling of the process R and therefore need only consider the case of equation (2.11) given in the statement of the lemma in the sequel. □

With these restrictions on the functions in the semigroup density, we are now in a position to prove the main result of this section. This proof is similar to the proof of Theorem 10 in the previous section and once again relies upon equating the Laplace transform of a function involving Φ with that of a Bessel function.

Theorem 12. *If R is a Markov process taking values on \mathbb{R} with a semigroup density of the form (2.2), then $2\rho(R_t)$ is equivalent in distribution to a squared-Bessel process. Thus, any Markov process satisfying the time inversion property and (H1-3) is in h -transform with a process with this equivalence in distribution.*

Proof. From Lemma 11, we know that, up to a time-scaling of R_t , ρ takes the form (2.11) for a constant $\sigma > 0$. That is, there exists a $c > 0$ such that $\{R_{ct}\}_{t \geq 0}$ has a semigroup density of the form (2.2) where ρ can be written (2.11). We first demonstrate the statement of the proof for this particular time-scaling and other time-scales follow using the same methodology.

By using the semigroup density (2.2) and the conservative property (H3) we know that for an $x > 0$

$$1 = \int_{\mathbb{R}} \Phi\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \frac{\theta(y)}{t^{\nu+1}} \exp\left(-\frac{\rho(x) + \rho(y)}{t}\right) dy.$$

Substituting the expression for ρ given above and the expression for θ given in (2.10) and splitting the integral into the positive and negative cases we have that

$$\begin{aligned} 1 &= \int_0^\infty \Phi\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) K_1 \frac{|y|^{2\nu+1}}{t^{\nu+1}} e^{-\frac{x^2+y^2}{2t}} dy + \int_{-\infty}^0 \Phi\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) K_2 \frac{|y|^{2\nu+1}}{t^{\nu+1}} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} dy \\ &= \int_0^\infty \Phi\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) K_1 \frac{|y|^{2\nu+1}}{t^{\nu+1}} e^{-\frac{x^2+y^2}{2t}} dy + \int_0^\infty \Phi\left(\frac{x}{\sqrt{t}}, -\frac{y}{\sqrt{t}}\right) K_2 \frac{y^{2\nu+1}}{t^{\nu+1}} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} dy, \end{aligned}$$

where we have let $y \rightarrow -y$ in the second integral.

If we now make the familiar substitutions $y = \frac{2\sqrt{u}t}{x}$ in the first integral and $y = \frac{2\sqrt{u}\sigma t}{x}$ in the second integral and set $\mu = \frac{2t}{x^2}$,

$$1 = \int_0^\infty (\Phi(1, 2\sqrt{u}) K_1 + \Phi(1, -2\sqrt{u}\sigma) K_2 \sigma^{2\nu+2}) u^\nu 2^\nu \mu^{\nu+1} e^{-\frac{1}{\mu}} e^{-\mu u} du,$$

which we note permits $\mu \in (0, \infty)$ for $t > 0$ and $x \in \mathbb{R}$. Some additional simplification of the integral and taking out terms not dependent on u yields a Laplace transform similar to that in Theorem 10:

$$\frac{e^{\frac{1}{\mu}}}{\mu^{\nu+1}} = \int_0^\infty (\Phi(1, 2\sqrt{u}) K_1 + \Phi(1, -2\sqrt{u}\sigma) K_2 \sigma^{2\nu+2}) u^\nu 2^\nu e^{-\mu u} du,$$

where the expression inside the brackets can be shown to be continuous on $(0, \infty)$ and therefore equivalent to a Bessel function by the injective property of the Laplace transform. Thus, we have

$$(\Phi(1, 2\sqrt{u}) K_1 + \Phi(1, -2\sqrt{u}\sigma) K_2 \sigma^{2\nu+2}) u^{\frac{\nu}{2}} 2^\nu = I_\nu(2\sqrt{u}), \quad (2.12)$$

for ν restricted to $\nu > -1$. We note that we have the same violation of assumption (H3) seen in Theorem 10 for $\nu \leq -1$.

Now, to determine the distribution of $2\rho(R_t)$, we evaluate the expectation for any bounded and Borel f and $x > 0$ of

$$\begin{aligned} \mathbb{E}_x[f(2\rho(R_t))] &= \int f(2\rho(y)) \Phi\left(x, \frac{y}{t}\right) \frac{\theta(y)}{t^{\nu+1}} \exp\left(-\frac{\rho(x) + \rho(y)}{t}\right) dy \\ &= \int_0^\infty f\left(\frac{4ut^2}{x^2}\right) \left[(\Phi(1, 2\sqrt{u}) K_1 + \Phi(1, -2\sqrt{u}\sigma) K_2 \sigma^{2\nu+2}) u^{\frac{\nu}{2}} 2^\nu \right] u^{\frac{\nu}{2}} \frac{2^{\nu+1} t^{\nu+1}}{x^{2\nu+2}} e^{-\frac{x^2}{2t}} e^{-\frac{2ut}{x^2}} du, \end{aligned}$$

where we have split this into its positive and negative parts and made the same substitutions. Then, by the representation of the modified Bessel function (2.12) and the expression for ρ given in (2.11),

$$\mathbb{E}_x[f(2\rho(R_t))] = \int_0^\infty f\left(\frac{2ut^2}{\rho(x)}\right) I_\nu(2\sqrt{u}) u^{\frac{\nu}{2}} \frac{t^{\nu+1}}{\rho(x)^{\nu+1}} e^{-\frac{\rho(x)}{t}} e^{-\frac{2ut}{x^2}} du.$$

If we now reverse the original substitution that we made to get u to return to our original

integrated variable y ,

$$\mathbb{E}_x [f(2\rho(R_t))] = \int_0^\infty f(y) I_\nu \left(\frac{\sqrt{2\rho(x)y}}{t} \right) \frac{y^{\frac{\nu}{2}}}{2\rho(x)^{\frac{\nu}{2}} t} e^{-\frac{2\rho(x)}{2t}} e^{-\frac{y}{2t}} dy = \mathbb{E}_{2\rho(x)} [f(Q_t)]$$

where Q is a squared Bessel process of index ν . A similar methodology provides the result for $x < 0$.

We have therefore shown that $2\rho(R.)$ and a squared Bessel process have the same law for any time $t > 0$ and so by the self-similar and Markov properties, we have that $2\rho(R.)$ is a squared Bessel process initiated at $2\rho(x)$ for R on the state space $\mathbb{R} \setminus \{0\}$. The continuity of the paths of the process are guaranteed by the continuity of the paths of the squared Bessel process. Thus, since the process $2\rho(R.)$ is self-similar, restricted to \mathbb{R}^+ and Markov with continuous paths, at zero it must have the unique entrance law associated with a squared Bessel process by Rivero [2005]. This implies that $2\rho(R.)$ is a squared Bessel process for R on the entirety of \mathbb{R} . \square

Remark 13. If $\rho \in C^2(\mathbb{R})$, then $\sigma = 1$ and $\rho(z) = \frac{z^2}{2\tilde{\sigma}^2}$ where $\tilde{\sigma}$ is the time-scaling of R . In this case Theorem 12 reduces to

$$\frac{\|R_t\|}{\tilde{\sigma}} \sim r_t$$

where $r := (r_t)_{t \geq 0}$ is a Bessel process on \mathbb{R}^+ initiated at $\sqrt{2\rho(x)}$.

This result leads us to introduce the notation $\hat{\rho}(x) = \sqrt{2\rho(x)}$ where ρ is the function in the semigroup density (2.2). This implies that for any Markov process R on \mathbb{R} with a semigroup density of the form (2.2), $\hat{\rho}(R_t)$ is a Bessel process.

In the sequel, we apply this result to processes killed at their first hitting time of zero and their extensions separately.

2.4.2 Characterisation for Processes Killed at Zero

Initially, we focus on processes on \mathbb{R} restricted to before their first hitting time of zero. Extensions of these processes are dealt with later.

If R is a Markov process on \mathbb{R} satisfying (H1-3) then we show, up to an h -transform and time-scaling, that the process only has the time inversion property if the process restricted to times less than its first hitting time of zero, which we call T_0 , can be written in the Lamperti-Kiu form given by

$$LK(B^{(\nu)}, \sigma B^{(\nu)}, \log \sigma, -\log \sigma, q^+, q^-). \quad (2.13)$$

Here, $B_t^{(\nu)} = B_t + \nu t$ for a standard Brownian motion B and $\nu > -1$, $\sigma > 0$ and $q^+, q^- \geq 0$. This allows us to find the semigroup density of the process explicitly.

We split the proof of this result into two separate lemmas. Firstly, we show that any Markov process on \mathbb{R} with a semigroup density of the form (2.2) satisfying (H1-3) can be

expressed as the Lamperti-Kiu representation given in (2.13) when it is restricted to the interval $(0, T_0)$. Thus, any Markov process on \mathbb{R} satisfying (H1-3) and the time inversion property can either be expressed using this representation or is in h -transform with such a process. The second of these lemmas calculates the semigroup density of the Lamperti-Kiu representation explicitly. Together with the results of Lawi [2008], the resulting form of this semigroup is enough to prove that the process has the time inversion property. This means we have our necessary and sufficient condition. For more details of the Lamperti-Kiu representation itself, we refer the reader to Section 1.1.4.

Lemma 14. *If R is a Feller process on \mathbb{R} satisfying (H1-3) and the time inversion property, then when R is restricted to the interval $(0, T_0)$, the process on $\mathbb{R} \setminus \{0\}$ can be written, up to an h -transform and time-scaling, in the Lamperti-Kiu form (2.13).*

We prove this result using two methods. The first, and more concise, uses the expression of the generator of the process given in Chaumont et al. [2013] and Volkonski's Theorem, see Volkonski [1958]. The second looks at the paths of the processes in a more intuitive way.

Proof Using Volkonski's Theorem. Firstly, if R enjoys the time inversion property, then up to an h -transform it has a semigroup density of the form (2.2). It is this version that we focus on noting that our results hold up to an h -transform. This semigroup density implies that R is a time homogeneous self-similar Markov process and therefore, by Chaumont et al. [2013], when it is restricted to $t < T_0$ it has a Lamperti-Kiu representation. Consequently, this proof focusses largely on deducing the possible parameters of the process.

Since we are working up to a time-scaling we only consider processes where ρ is of the form in (2.11). R restricted to $t < T_0$ has the Lamperti-Kiu representation so its generator applied to a function $f \in C_0^2(\mathbb{R} \setminus \{0\})$ has the form

$$\mathcal{A}_R f(x) = \frac{b^\pm}{x} f'(x) + \frac{(a^\pm)^2}{2} f''(x) \quad (2.14)$$

$$+ \frac{q^\pm}{x^2} (\mathbb{E} [f(-x \exp(U^\pm))] - f(x)) \quad (2.15)$$

$$+ \frac{1}{x^2} \int_0^\infty (f(xu) - f(x) - x f'(x) l(\log u)) \Theta^\pm(du) \quad (2.16)$$

by [Chaumont et al., 2013, Proposition 7]. Moreover, applying Volkonski's Theorem, see Volkonski [1958], using the time-change A_t (the inverse of $H_t = \int_0^t \hat{\rho}(R_s)^{-2} ds$)

$$\begin{aligned} \mathcal{A}_{R_A} f(x) &= \frac{1}{\sigma^2(\text{sgn}(x))} \left[b^\pm x f'(x) + \frac{(a^\pm)^2}{2} x^2 f''(x) \right. \\ &\quad + q^\pm (\mathbb{E} [f(-x \exp(U^\pm))] - f(x)) \\ &\quad \left. + \int_0^\infty (f(xu) - f(x) - x f'(x) l(\log u)) \Theta^\pm(du) \right] \end{aligned}$$

This is a Feller multiplicative process and therefore has the representation given in [Chaumont

et al., 2013, Section 2.2]. If we consider the positive case, then the result in Chaumont implies

$$\begin{aligned}\mathcal{A}_{\xi^+}f(x) &= b^\pm x f'(x) + \frac{(a^\pm)^2}{2} x^2 f''(x) \\ &\quad + \int_0^\infty (f(xu) - f(x) - x f'(x) l(\log u)) \Theta^\pm(du).\end{aligned}$$

Furthermore, since $\hat{\rho}(R_t)$ is a Bessel process, $\hat{\rho}(R_{A_t})$ is a geometric Brownian motion. Since $\hat{\rho}(x) = x$ for $x > 0$, we have that ξ_+ is a Brownian motion with drift. The negative case follow similarly.

Finally, if we compare this with the expression of a Feller multiplicative process given in (1.2),

$$e^{B_t + \nu t} = x \exp \left(\xi_{t - T_{N_t^{(x)}}^{(x)}} + \sum_{k=0}^{N_t^{(x)} - 1} \left(\xi_{\zeta^{(x,k)}}^{(x,k)} + U^{(x,k)} \right) \right) \hat{\rho} \left(\exp \left(i\pi N_t^{(x)} \right) \right)$$

we have $U^- = \log \sigma$ and $U^+ = -\log \sigma$, for a $\sigma \in \mathbb{R}^+$.

□

Intuitive Proof. In this proof, we only deal with processes with a semigroup density of the form (2.2). As other processes with the time inversion property are in h -transform with these processes we simply note that the result holds up to an h -transform.

On the one hand, if the process has continuous paths, then the stopping property implies that zero is never crossed and the problem reduces to the case on \mathbb{R}^+ (or equivalently \mathbb{R}^-) characterised in Theorem 10. Namely, R is a Bessel process in the wide sense. This is given in the above Lamperti-Kiu representation when the relevant jumping parameter q^+ or q^- vanishes leaving a single time-changed exponential Lévy process. The choice of vanishing parameter depends solely upon the sign of the starting position; q^+ if we start at a positive point and vice-versa. The ξ^+ and ξ^- given in the statement of the theorem provide the time-scaled Bessel processes we require to construct a Bessel process through the Lamperti transform for positive self-similar Markov processes, see Lamperti [1972].

Considering the case with jumps, the first thing we note is that a semigroup density of this form implies self-similarity. Coupled with the stopping and Feller properties, the result of Chaumont et al. [2013] states that the process must therefore enjoy a Lamperti-Kiu representation. Specifically, it can be expressed as $LK(\xi^+, \xi^-, U^+, U^-, q^+, q^-)$ for Lévy processes ξ^+ and ξ^- and real-valued random variables U^+ and U^- . The remainder of this proof hinges on the deduction of these parameter. We do this by matching the independent jump and continuous stochastic processes of $\hat{\rho}(R.)$ and r on intervals between changes of sign.

As a starting point, Theorem 12 states that $\hat{\rho}(R_t^{(x)})$ is necessarily a Bessel process, which we denote $r_t^{(\hat{\rho}(x))}$. We take this opportunity to introduce the following time-change notation for

the Bessel process $r_t^{(x)}$ and our process $R_t^{(x)}$

$$\tilde{H}_t^{(x)} = \int_0^t \frac{1}{r_u^{(x)2}} du \quad \text{and} \quad H_t^{(x)} = \int_0^t \frac{1}{R_u^{(x)2}} du.$$

We also define the time changes from each jumping (or change of sign) time onwards for $H_t > T_n$ as

$$\tilde{H}_{T_n,t}^{(x)} = \int_{\tilde{A}_{T_n}}^t \frac{1}{r_u^{(x)2}} du \quad \text{and} \quad H_t^{(x)} = \int_{A_{T_n}}^t \frac{1}{R_u^{(x)2}} du.$$

where A_t is the inverse of the time change H_t . This indicates the time change started at the change of sign. That is, the time change started at the stopping time $\tilde{H}_{T_n,t}^{(x)} = H_t^{(x)} - T_n$.

The crossing times of zero are given by the times T_i that indicate the changes of sign. Since the process is stopped at zero, these jumps determining changes of sign all have strictly positive size. Thus, there is a strictly positive time between any such jumps at times T_i that involve a change in sign almost surely.

The relationship between the Bessel process r and R holds for all $x \in \mathbb{R} \setminus \{0\}$ and $t > 0$ so we first consider $x = 1 > 0$ and $H_t < T_1$ for $\rho(x)$ given in (2.11)

$$r_t^{(1)} = \hat{\rho}(R_t^{(1)}).$$

Using the Lamperti representation of the Bessel process and the Lamperti-Kiu representation for R , noting $N_{H_t} = 0$ when $H_t < T_1$, this gives

$$\exp\left(B_{\tilde{H}_t^{(1)}} + \nu \tilde{H}_t^{(1)}\right) = \exp\left(\xi_{H_t^{(1)}}^+\right),$$

where we have used that $\hat{\rho}(\lambda) = \lambda$ for $\lambda > 0$. Moreover, for $t < T_1$ or before the first jump to a negative point $\tilde{H}_t^{(1)} = H_t^{(1)}$ and so we have the representation of our first process

$$\xi_t^+ = B_t + \nu t.$$

Examining the case where $H_t \in (T_1, T_2)$, or equivalently $N_{H_t} = 1$, yields

$$\begin{aligned} \exp\left(B_{\tilde{H}_t^{(1)}} + \nu \tilde{H}_t^{(1)}\right) &= \hat{\rho}\left(\exp\left(\xi_{H_{T_1,t}^{(1)}}^- + \xi_{\tilde{H}_{T_1}} + U^+ + i\pi N_{H_t}^{(1)}\right)\right) \\ &= \exp\left(\xi_{H_{T_1,t}^{(1)}}^- + B_{\tilde{H}_{T_1}} + \nu \tilde{H}_{T_1} + U^+\right) \hat{\rho}(-1) \\ &= \exp\left(\xi_{H_{T_1,t}^{(1)}}^- + B_{\tilde{H}_{T_1}} + \nu \tilde{H}_{T_1} + U^+ - \log \sigma\right). \end{aligned}$$

This leaves $U^+ = \log \sigma$ as a constant and by the additive property of the time change

$$B_{\tilde{H}_t^{(1)}} + \nu \tilde{H}_t^{(1)} = \xi_{H_t^{(1)}}^-.$$

For $H_t \in (T_1, T_2)$, $\tilde{H}_{T_1, t}^{(1)} = \sigma^2 H_{T_1, t}^{(1)}$. This can be seen because $R_t = -\sigma r_t$ in this interval, by $\hat{\rho}(R_t) = r_t$ restricted to R_t negative and therefore

$$H_{T_1, t}^{(1)} = \int_{A_{T_1}}^t \frac{1}{R_u^{(1)2}} du = \sigma^{-2} \int_{A_{T_1}}^t \frac{1}{r_u^{(1)2}} du = \sigma^{-2} \tilde{H}_{T_1, t}^{(1)}.$$

If we now set a function $Z_t = \xi_{\sigma^2 t}^-$, this gives $Z_t = B_t + \nu t$ and so

$$\xi_u^- = B_{\sigma^2 u} + \nu \sigma^2 u.$$

Finally, considering $t \in (T_2, T_3)$

$$\exp \left(B_{\tilde{H}_t^{(1)}} + \nu \tilde{H}_t^{(1)} \right) = \exp \left(\xi_{H_{T_2, t}^{(1)}}^+ + \xi_{\sigma^{-2} \tilde{H}_{T_2}}^- + \xi_{\tilde{H}_{T_1}}^+ + U^- + \log \sigma \right)$$

leaves $U^- = -\log \sigma$.

To check these statements, we also consider the case when $x < 0$. The self-similarity of the process means that we only need to check a single starting point; we choose $x = -\sigma$ for simplicity. Once again, employing Theorem 12, $r_t^{(1)} = \hat{\rho}(R_t^{(-\sigma)})$ and the Lamperti and Lamperti-Kiu representations of r and R respectively

$$\exp \left(B_{\tilde{H}_t} + \nu \tilde{H}_t \right) = \hat{\rho} \left(-\sigma \exp \left(\xi_{H_{T_{N_t}, t}^{(\sigma)}}^{(N_t)} + \sum_{k=0}^{N_{H_t}^{(k)} - 1} (\xi_{\zeta^{(k+1)}}^{(x, k)} + U^{(x, k)}) + i\pi N_{H_t}^{(\sigma)} \right) \right).$$

For $H_t < T_1$ and $N_{H_t} = 0$

$$\exp \left(B_{\tilde{H}_t^{(1)}} + \nu \tilde{H}_t^{(1)} \right) = \hat{\rho}(-\sigma) \exp \left(\xi_{H_t^{(\sigma)}}^- \right).$$

Using $H_t^{(\sigma)} = \sigma^{-2} \tilde{H}_t^{(1)}$ the same result follows. □

In order to show that processes of the form (2.13) have the time inversion property, we first calculate the semigroup density of the process. Since we show that the semigroup density has the same form as that given in Lawi [2008], the process has the time inversion property and thus, we have our equivalence.

In the usual construction of a Lampert-Kiu representation, we time-change a Feller multiplicative process of the form (1.2) and so this process can be time-changed to return to the Feller multiplicative process. However, to compute the semigroup density explicitly, we time change the Lamperti-Kiu process using a slightly different additive process that results as the

inverse of the Hartman-Watson distribution, see Hartman and Watson [1974]. We see that this process is still a Feller multiplicative process and because the time-change is well known we can compute the semigroup density explicitly.

Lemma 15. *If R has the Lamperti-Kiu representation given by (2.13), then the semigroup density is given by*

$$p_t(x, y) = \begin{cases} \left[\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{xy}{t} \right) + \pi^- I_{|\nu|} \left(\frac{xy}{t} \right) \right] \frac{y^{\nu+1}}{x^{\nu} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} & \text{if } x > 0, y > 0; \\ \left[-\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{-xy}{\sigma t} \right) + \pi^+ I_{|\nu|} \left(\frac{xy}{\sigma t} \right) \right] \frac{|y|^{\nu+1}}{x^{\nu} \sigma^{\nu+2} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x > 0, y < 0; \\ \left[\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{-xy}{\sigma t} \right) + \pi^- I_{|\nu|} \left(\frac{-xy}{\sigma t} \right) \right] \frac{y^{\nu+1}}{|x|^{\nu} t} \sigma^{\nu} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2t}} & \text{if } x < 0, y > 0; \\ \left[-\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{xy}{\sigma^2 t} \right) + \pi^+ I_{|\nu|} \left(\frac{xy}{\sigma^2 t} \right) \right] \frac{|y|^{\nu+1}}{|x|^{\nu} \sigma^2 t} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x < 0, y < 0. \end{cases}$$

where $\pi^+ = \frac{q^+}{q^+ + q^-}$ and $\pi^- = \frac{q^-}{q^+ + q^-}$. Thus, in the case where the process avoids zero almost surely, this process and any time-scaling or h -transform of this process has the time inversion property by Lawi [2008]. In the case of continuous paths this reduces to the semigroup density of a Bessel process restricted to either \mathbb{R}^+ or \mathbb{R}^- given in the previous section.

Proof. Firstly, we define a Feller multiplicative invariant process $Z := (Z_t)_{t \geq 0}$ on $\mathbb{R} \setminus \{0\}$ (for which 0 is polar) with infinitesimal generator \mathcal{A}_Z applied to $f \in C_0^2(\mathbb{R})$ that vanishes along with its derivatives at zero given by

$$\mathcal{A}_Z f(x) = \frac{x^2}{2} f''(x) + \frac{2\nu + 1}{2} x f'(x) + q(\text{sgn}(x)) (f(-\kappa(\text{sgn}(x))x) - f(x))$$

for $\nu > 0$ and

$$\kappa(\text{sgn}(x)) = \begin{cases} \sigma & \text{if } x > 0; \\ \frac{1}{\sigma} & \text{if } x < 0. \end{cases}$$

By Chaumont et al. [2013], a process with a generator of this form also has the representation

$$Z_t^{(x)} = \sigma(\text{sgn}(x)) \exp \left(B_t + \nu t + \sum_{k=0}^{N_t^{(x)} - 1} U^{(\text{sgn}(x), k)} + i\pi N_t^{(x)} \right)$$

where $(N_t^{(x)}, t \geq 0)$ is an alternating renewal process started at 0 with rates q^+ and q^- . Moreover, from the form of the generator in Chaumont et al. [2013], we can also deduce that $U^{(\text{sgn}(x), k)}$ is given by

$$U^{(\text{sgn}(x), k)} = \begin{cases} \log \sigma & \text{if } \text{sgn}(x)(-1)^k = 1; \\ -\log \sigma & \text{if } \text{sgn}(x)(-1)^k = -1, \end{cases}$$

and

$$\sigma(\operatorname{sgn}(x)) = \begin{cases} 1 & \text{if } x > 0 \\ -\sigma & \text{if } x < 0. \end{cases}$$

Let $J_t^{(x)} := e^{-B_t - \nu t} Z_t^{(x)}$, $t \geq 0$ and we can see that this process has the property $\hat{\rho}(J_t^{(x)}) = 1$ for $t \geq 0$ by the form of $\hat{\rho}$. Furthermore, $(J_t^{(x)}, t \geq 0)$ has only a jumping stochastic part that jumps from positive to negative with its absolute value determined by $\sigma(\operatorname{sgn}(x))$. It is therefore a two dimensional Markov chain on $\{1, -\sigma\}$ with generator matrix

$$Q = \begin{pmatrix} -q^+ & q^+ \\ q^- & -q^- \end{pmatrix}$$

and, by [Ross, 2010, p. 404], the transition probabilities are given by

$$\begin{aligned} \mathbb{P}_1(J_t = 1) &= \pi^+ e^{-qt} + \pi^- & \mathbb{P}_1(J_t = -\sigma) &= \pi^+(1 - e^{-qt}) \\ \mathbb{P}_{-\sigma}(J_t = 1) &= \pi^-(1 - e^{-qt}) & \mathbb{P}_{-\sigma}(J_t = -\sigma) &= \pi^- e^{-qt} + \pi^+. \end{aligned}$$

where

$$q = q_+ + q_-, \quad \pi_{\pm} = \frac{q^{\pm}}{q}$$

By the Lamperti-Kiu representation in Chaumont et al. [2013], we have that the Lamperti-Kiu process R can be written as a time-change of the Feller multiplicative process Z

$$(R_t, t \geq 0 | R_0 = \sigma(\operatorname{sgn}(x))) \stackrel{(d)}{=} \left(Z_{\gamma_t^{(\nu)}}^{(x)}, t \geq 0 \right) \quad (2.17)$$

where the time change is given by

$$\gamma_t^{(\nu)} = \inf\{s \geq 0 : \int_0^s \hat{\rho}(Z_u)^2 du > t\}, \quad t \geq 0.$$

Thus, we can write the Laplace-Mellin transform of the process

$$\begin{aligned} A_x(\lambda, \beta) &:= \int_0^\infty e^{-\lambda u} \mathbb{E}_{\sigma(\operatorname{sgn}(x))} [R_u^{-\beta}] du \\ &= \mathbb{E} \left[\int_0^\infty e^{-\lambda \hat{A}_t^{(\nu)}} (Z_t^{(x)})^{-\beta} d\hat{A}_t^{(\nu)} \right] \\ &= \int_0^\infty \mathbb{E} \left[e^{-\lambda \hat{A}_t^{(\nu)} + (2-\beta) B_t^{(\nu)}} \right] \mathbb{E} \left[(J_t^{(x)})^{-\beta} \right] dt. \end{aligned}$$

Now, recall the following formula from [Matsumoto and Yor, 2005a, Theorem 4.11, p. 325]. We have

$$\int_0^\infty e^{-\theta t} \mathbb{P} \left(e^{B_t^{(\nu)}} \in dy, A_t^{(\nu)} \in du \right) dt = y^{-\mu + \nu - 2} p^{(\mu)}(u, 1, y) dy du \quad (2.18)$$

for $\nu, \theta, y, u > 0$ where $\mu = \sqrt{2\theta + \nu^2}$ and $p^{(\mu)}(u, x, y) dy$ is the semigroup of a BES(ν). Notice

that $\mu = |\nu|$ when $\theta = 0$ which leads to

$$\int_0^\infty \mathbb{P} \left(e^{B_t^{(\nu)}} \in dy, A_t^{(\nu)} \in du \right) dt = y^{2\nu-2} p^{(|\nu|)}(u, 1, y) dy du \quad (2.19)$$

Applying (2.18), with $\mu = \sqrt{\nu^2 + 2q}$, we obtain

$$\begin{aligned} A_1(\lambda, \beta) &= \int_0^\infty \mathbb{E} \left[e^{-\lambda \hat{A}_t^{(\nu)} + (2-\beta) B_t^{(\nu)}} \right] (\pi_+ e^{-qt} + \pi_-) dt + \int_0^\infty \mathbb{E} \left[e^{-\lambda \hat{A}_t^{(\nu)} + (2-\beta) B_t^{(\nu)}} \right] (-\sigma)^{-\beta} \pi_+ (1 - e^{-qt}) dt \\ &= \int_0^\infty du e^{-\lambda u} \int_0^\infty dy y^{2-\beta} \left\{ \pi_+ y^{-\mu+\nu-2} p^{(\mu)}(u, 1, y) + \pi_- y^{2\nu-2} p^{(|\nu|)}(u, 1, y) \right\} \\ &\quad + \int_0^\infty du e^{-\lambda u} \int_0^\infty dy y^{2-\beta} (-\sigma)^{-\beta} \left\{ \pi_+ \left(-y^{-\mu+\nu-2} p^{(\mu)}(u, 1, y) + y^{2\nu-2} p^{(|\nu|)}(u, 1, y) \right) \right\} \end{aligned}$$

and a further simplification yields,

$$\begin{aligned} A_1(\lambda, \beta) &= \int_0^\infty du e^{-\lambda u} \int_0^\infty dy y^{-\beta} \left\{ \pi_+ y^{-\mu+\nu} p^{(\mu)}(u, 1, y) + \pi_- y^{2\nu} p^{(|\nu|)}(u, 1, y) \right\} \\ &\quad + \int_0^\infty du e^{-\lambda u} \int_0^\infty dy (-\sigma y)^{-\beta} \left\{ \pi_+ (-y^{-\mu+\nu} p^{(\mu)}(u, 1, y) + y^{2\nu} p^{(|\nu|)}(u, 1, y)) \right\} \\ &= \int_0^\infty du e^{-\lambda u} \int_0^\infty dy y^{-\beta} \left\{ \pi_+ y^{-\mu+\nu} p^{(\mu)}(u, 1, y) + \pi_- y^{2\nu} p^{(|\nu|)}(u, 1, y) \right\} \\ &\quad + \int_0^\infty du e^{-\lambda u} \int_{-\infty}^0 \frac{dz}{\sigma} z^{-\beta} \left\{ \pi_+ \left(-(-z/\sigma)^{-\mu+\nu} y^{2\nu} p^{(|\mu|)}(u, 1, -z/\sigma) + y^{2\nu} p^{(|\nu|)}(u, 1, -z/\sigma) \right) \right\} \end{aligned}$$

Since we have

$$A_1(\lambda, \beta) = \int_0^\infty e^{-\lambda u} \int y^{-\beta} p_u(1, y) dy,$$

inverting this Laplace-Mellin transform yields

$$p_t(1, y) = \begin{cases} \left(\pi_+ I_{\sqrt{\nu^2+2q}} \left(\frac{y}{t} \right) + \pi_- I_{|\nu|} \left(\frac{y}{t} \right) \right) y^{\nu+1} t^{-1} e^{-\frac{1}{2t}} e^{-\frac{y^2}{2t}} & \text{if } y > 0; \\ \pi_+ \left(-I_{\sqrt{\nu^2+2q}} \left(\frac{-y}{\sigma t} \right) + I_{|\nu|} \left(\frac{-y}{\sigma t} \right) \right) (-y)^{\nu+1} \sigma^{-\nu-2} t^{-1} e^{-\frac{1}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } y < 0. \end{cases}$$

If we do the same for the negative case initiated at $-\sigma$

$$\begin{aligned} A_{-\sigma}(\lambda, \beta) &:= \int_0^\infty e^{-\lambda u} \mathbb{E}_{-\sigma} \left[R_u^{-\beta} \right] du \\ &= \mathbb{E} \left[\int_0^\infty e^{-\lambda \hat{A}_t^{(\nu)}} (Z_t^{(-\sigma)})^{-\beta} d(\hat{A}_t^{(\nu)}) \right] \\ &= \int_0^\infty \mathbb{E} \left[e^{-\lambda \hat{A}_t^{(\nu)} + (2-\beta) B_t^{(\nu)}} \right] \mathbb{E} \left[(J_t^{(-\sigma)})^{-\beta} \right] dt, \end{aligned}$$

which gives

$$\begin{aligned}
A_{-\sigma}(\lambda, \beta) &= \int_0^\infty \mathbb{E} \left[e^{-\lambda \hat{A}_t^{(\nu)} + (2-\beta) B_t^{(\nu)}} \right] (-\sigma)^{-\beta} (\pi_- e^{-qt} + \pi_+) dt \\
&\quad + \int_0^\infty \mathbb{E} \left[e^{-\lambda \hat{A}_t^{(\nu)} + (2-\beta) B_t^{(\nu)}} \right] \pi_- (1 - e^{-qt}) dt \\
&= \int_0^\infty du e^{-\lambda u} \int_{-\infty}^0 \frac{dy}{\sigma} y^{-\beta} \left\{ \pi_- (-y/\sigma)^{-\mu+\nu} p^{(\mu)}(u, 1, -y/\sigma) + \pi_+ (-y/\sigma)^{2\nu} p^{(|\nu|)}(u, 1, -y/\sigma) \right\} \\
&\quad + \int_0^\infty du e^{-\lambda u} \int_0^\infty dy y^{-\beta} \left\{ \pi_- \left(-y^{-\mu+\nu} p^{(\mu)}(u, 1, y) + y^{2\nu} p^{(|\nu|)}(u, 1, y) \right) \right\}.
\end{aligned}$$

Since we have

$$A_{-\sigma}(\lambda, \beta) = \int_0^\infty e^{-\lambda u} \int y^{-\beta} p_u(-\sigma, y) dy,$$

inverting this Laplace-Mellin transform yields

$$p_t(-\sigma, y) = \begin{cases} \pi_- \left(-I_{\sqrt{\nu^2+2q}} \left(\frac{y}{t} \right) + I_{|\nu|} \left(\frac{y}{t} \right) \right) y^{\nu+1} t^{-1} e^{-\frac{1}{2t}} e^{-\frac{y^2}{2t}} & \text{if } y > 0; \\ \left(\pi_- I_{\sqrt{\nu^2+2q}} \left(\frac{-y}{\sigma t} \right) + \pi_+ I_{|\nu|} \left(\frac{-y}{\sigma t} \right) \right) (-y)^{\nu+1} \sigma^{-\nu-2} t^{-1} e^{-\frac{1}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } y < 0. \end{cases}$$

The self-similarity permits us to generalise these semigroup densities to any starting point and so implies that

$$p_t(x, y) = \begin{cases} x^{-1} p_{t/x^2}(1, y/x) & \text{if } x > 0; \\ (\sigma/|x|) p_{(t\sigma^2)/x^2}(-\sigma, \sigma y/|x|) & \text{if } x < 0. \end{cases}$$

Finally, this gives the semigroup density given in the statement of the lemma. □

We combine Lemma 14 and Lemma 15 together to produce a sufficient and necessary condition for the time inversion property on \mathbb{R} .

Theorem 16. *Let R be a Markov process on \mathbb{R} satisfying (H1-3) and killed at the first time it hits zero T_0 . If R has the time inversion property then R restricted to $t < T_0$ has the Lamperti-Kiu representation given by (2.13). Here, $\nu > -1$, $\sigma > 0$ and $q^+, q^- \geq 0$. We have a Bessel process (or negative Bessel process) in the case where $q^+ = q^- = 0$.*

Furthermore, if R is not a Bessel process or a negative Bessel process, then the semigroup

density of this Lampert-Kiu representation is given by

$$p_t(x, y) = \begin{cases} \left[\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{xy}{t} \right) + \pi^- I_{|\nu|} \left(\frac{xy}{t} \right) \right] \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} & \text{if } x > 0, y > 0; \\ \left[-\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{-xy}{\sigma t} \right) + \pi^+ I_{|\nu|} \left(\frac{xy}{\sigma t} \right) \right] \frac{y^{\nu+1}}{x^\nu \sigma^{\nu+2} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x > 0, y < 0; \\ \left[\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{-xy}{\sigma t} \right) + \pi^- I_{|\nu|} \left(\frac{-xy}{\sigma t} \right) \right] \frac{y^{\nu+1}}{x^\nu t} \sigma^\nu e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2t}} & \text{if } x < 0, y > 0; \\ \left[-\pi^+ I_{\sqrt{\nu^2 + 2(q^+ + q^-)}} \left(\frac{xy}{\sigma^2 t} \right) + \pi^+ I_{|\nu|} \left(\frac{xy}{\sigma^2 t} \right) \right] \frac{y^{\nu+1}}{x^\nu \sigma^2 t} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x < 0, y < 0, \end{cases}$$

where $\pi^+ = \frac{q^+}{q^+ + q^-}$ and $\pi^- = \frac{q^-}{q^+ + q^-}$. This form of the semigroup density implies that R with the Lampert-Kiu form given by (2.13) has the time inversion property provided it does not hit zero almost surely.

The Bessel process stopped at zero is included in this representation as the degenerate case where either q^+ or q^- vanishes for the Bessel process on the positive half-line or negative half-line respectively.

This Theorem coupled with Theorem 12 means that, up to an h -transform and a time-scaling, any process satisfying (H1-3) and the time inversion property has the representation (2.13) when restricted to $t < T_0$. Moreover, we note that this semigroup density guarantees the time inversion property in the case where the process avoids zero almost surely by the results of Lawi [2008].

Proof of Theorem 16. If R is a Markov process on \mathbb{R} satisfying (H1-3), we can show that, up to an h -transform, the time inversion property implies it can be expressed as the Lampert-Kiu representation (2.13) using Lemma 14. Furthermore, the semigroup density follows from Lemma 15 and from this we can see that when a process of this form avoids zero almost surely, it has the time inversion property. \square

2.4.3 Characterisation of Recurrent Extensions

Now that we have fully characterised all processes enjoying the time inversion property up to their first hitting time of zero, we consider possible extensions of these processes that do not violate the time inversion or Markov properties. We do this by determining all possible entrance laws of the original process which we call the minimal process. More explicitly, assume that we are given a time homogeneous Markov process $\hat{X} := \{\hat{X}_t\}_{t \geq 0}$ on state space E that hits a fixed point $b \in S$ in finite time almost surely and such that b is a trap, \hat{X} remains at b almost surely once b has been reached. \hat{X} is our minimal process. A recurrent extension of the minimal process, is a Markov process $X := \{X_t\}_{t \geq 0}$ such that

$$\mathbb{P}_x(X_t \in dy; t < T_b) = \mathbb{P}_x(\hat{X}_t \in dy; t < T_b)$$

for all $x, y \in E$, where T_b is the first hitting time of b and b is regular for $\{b\}$ relative to X . The recurrence of X_t then follows from the property above and the Markov and time homogeneous

properties. For more details we refer the reader to Blumenthal [1992]. In our particular case, we consider extensions from the point 0 and only consider recurrent extensions that also have the time inversion property.

Once we have these extensions, we can characterise the processes explicitly through their semigroup density, which, in turn, allows us to show that these processes are guaranteed to have the time inversion property. This leads to the following result.

Theorem 17. *Let $\hat{R} := (\hat{R}_t)_{t \geq 0}$ be a minimal Markov process on \mathbb{R} satisfying (H1-3), which is stopped at zero, and let R be its recurrent extension.*

- (i) *If \hat{R} , and therefore R , have continuous paths almost surely, then R has the time inversion property if and only if, up to an h -transform, it is a skew Bessel process with a possible change in time-scaling when the process changes sign. The semigroup density of the process is given by*

$$p_t(x, y) = \begin{cases} [\beta I_\nu(\frac{xy}{t}) + (1 - \beta) I_{-\nu}(\frac{xy}{t})] \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} & \text{if } x \geq 0, y \geq 0 \\ \beta (I_{-\nu}(-\frac{xy}{\sigma t}) - I_\nu(-\frac{xy}{\sigma t})) \frac{|y|^{\nu+1}}{x^\nu \sigma^{\nu+2} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x \geq 0, y < 0 \\ (1 - \beta) (I_{-\nu}(-\frac{xy}{\sigma t}) - I_\nu(-\frac{xy}{\sigma t})) \frac{y^{\nu+1} \sigma^\nu}{|x|^\nu t} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2t}} & \text{if } x < 0, y \geq 0 \\ [(1 - \beta) I_\nu(\frac{xy}{\sigma^2 t}) + \beta I_{-\nu}(\frac{xy}{\sigma^2 t})] \frac{|y|^{\nu+1}}{|x|^\nu \sigma^2 t} e^{-\frac{x^2+y^2}{2\sigma^2 t}} & \text{if } x < 0, y < 0 \end{cases}.$$

for a $\beta \in [0, 1]$, $\sigma > 0$ and $\nu \in (-1, 0)$ up to a time-scaling. Furthermore, from this we can show that $\lim_{x \rightarrow 0} \mathbb{P}_x(R_t > 0) = \beta$ for all $t > 0$.

- (ii) *If \hat{R} has a non-zero probability of jumping, then R has the time inversion property if and only if, up to an h -transform and time-scaling, it has the semigroup density*

$$p_t(x, y) = \begin{cases} [\pi^+ I_{\sqrt{\nu^2+2q}}(\frac{xy}{t}) + \pi^- I_\nu(\frac{xy}{t})] \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} & \text{if } x \geq 0, y \geq 0 \\ [-\pi^+ I_{\sqrt{\nu^2+2q}}(\frac{-xy}{\sigma t}) + \pi^+ I_\nu(\frac{-xy}{\sigma t})] \frac{|y|^{\nu+1}}{x^\nu \sigma^{\nu+2} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x \geq 0, y < 0 \\ [-\pi^- I_{\sqrt{\nu^2+2q}}(\frac{-xy}{\sigma t}) + \pi^- I_\nu(\frac{-xy}{\sigma t})] \frac{y^{\nu+1} \sigma^\nu}{|x|^\nu t} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2t}} & \text{if } x < 0, y \geq 0 \\ [-\pi^+ I_{\sqrt{\nu^2+2q}}(\frac{xy}{\sigma^2 t}) + \pi^+ I_\nu(\frac{xy}{\sigma^2 t})] \frac{|y|^{\nu+1}}{|x|^\nu \sigma^2 t} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x < 0, y < 0, \end{cases} \quad (2.20)$$

where $\pi^\pm = \frac{q^\pm}{q^+ + q^-}$, $q^+ + q^- = q$ for $q^+, q^- \geq 0$ (at least one strictly greater than zero), $\sigma > 0$ and $\nu \in (-1, 0)$. Here, we have $\lim_{x \rightarrow 0} \mathbb{P}_x(R_t > 0) = \pi^-$ for all $t > 0$.

Remark 18. *It can be seen that case (ii) reduces to case (i) by taking $\pi^+ = (1 - \beta)$ in the limit as q^+ and q^- tend to zero. However, because π^+ is uniquely determined by the jumping coefficients, we note that there is greater flexibility in our choice of parameters in case (i), the case of almost surely continuous paths.*

In the remainder of this section we prove this result. Firstly, we consider the extensions of processes with jumps before returning to the case of continuous paths. We initially use an excursion proof for both cases by considering all the possible entrance laws that do not violate the Markov or self-similar properties. However, for the continuous paths, we also consider a second proof of the semigroup density that extends the approach of Blei [2012] for the skew Bessel process and uses PDEs.

In order to uniquely determine the recurrent extensions of processes with the time inversion property, we shall uniquely determine the entrance laws. By Theorem 12, this will be enough to uniquely determine the extension.

Remark 19. *Following on from Remark 7, we highlight here that we only use the restriction on the semigroup density (2.2) on $\mathbb{R} \setminus \{0\}$ and therefore, we use the slightly weaker assumption (H2). This permits us to include processes whose semigroup densities are not continuous at the origin.*

For this reason, this section considers all possible recurrent extensions of the stopped processes in Theorem 16, whose semigroup density satisfies (2.2) on $\mathbb{R} \setminus \{0\}$.

Proof of Theorem 17 Part II: The jump case

If the process R has a non-zero probability of jumps in its state space S prior to its first hitting time of zero, then, by the Markov property, its extension also bears this property. Consequently, Theorem 16 implies that we only need consider extensions where the minimal process has the Lampert-Kiu representation (2.13) for one or both of q^+, q^- strictly greater than zero. In order to determine the process uniquely, we shall determine its entrance law explicitly in terms of the entrance law of a Bessel process with the same index as the Bessel process given by $\hat{\rho}(R_t)$. Moreover, the absolute continuity of the semigroup with respect to the Lebesgue measure and Theorem 12 implies that the process leaves zero immediately and continuously. We note that since the process leaves zero continuously, by Rogers [1983], the entrance law is sufficient to uniquely determine the process.

Remark 20. *Applying Theorem 12 and the time inversion property provides a second way of seeing the almost sure continuity of the paths at zero. Theorem 12 implies that neither the process nor its h -transforms can jump to zero, therefore, as the time inversion of the process also has the time inversion property, it also cannot jump to zero. As time inversion involves inverting the time variable, and the process itself is an inverted version of another process with the time inversion property, the process must leave zero continuously.*

We begin by showing existence of the entrance law followed by showing its uniqueness. Finally, we use this entrance law to calculate the semigroup density explicitly.

Lemma 21 (Existence of the Entrance Law). *Let \hat{R} be a minimal Feller process on \mathbb{R} that hits zero almost surely, is stopped at its first hitting time of zero, and which can be expressed as the Lamperti-Kiu representation given in (2.13). There exists an extension of \hat{R} , which is*

determined by the entrance law

$$\tilde{\eta}_t(dx) = \pi^- \mathbf{1}_{[x>0]} n_t(dx) + \pi^+ \mathbf{1}_{[x<0]} \bar{n}_t^\sigma(dx)$$

where $n_t(dx)$ is the entrance law of the extended Bessel process r on \mathbb{R}^+ given in Rivero [2005] and $\pi^\pm = \frac{q^\pm}{q^\pm + q^\mp}$. Moreover, $\bar{n}_t^\sigma(dx) = n_t(dy) |_{y=-x/\sigma}$.

Proof. By [Blumenthal, 1992, p. 137 Chapter V Section 2.b], we know that extensions of Markov processes, up to a constant multiple, are given by entrance law measures that satisfy the following equation

$$\int \int f(y) p_s^0(x, dy) \tilde{\eta}_t(dx) = \int f(y) \tilde{\eta}_{t+s}(dy), \quad (2.21)$$

where p^0 denotes the semigroup density of the process killed at the first hitting time at zero. In this proof, we show that the entrance law in the statement of the lemma satisfies this equation for the minimal process given by the Lamperti-Kiu representation (2.13).

The left hand side of the equation is given by

$$\int \int \tilde{\eta}_t(dx) f(y) p_s^0(x, dy) = \int \tilde{\eta}_s(dx) \mathbb{E}_x [f(R_s); t < T_0].$$

From Theorem 12, R_t satisfies $\hat{\rho}(R_t) = r_t$. Thus, R_t is given by r_t when R_t is positive and $-\sigma r_t$ when R_t is negative. Applying this to the expectation yields

$$\begin{aligned} & \int \int \tilde{\eta}_t(dx) f(y) p_s^0(x, dy) \\ &= \int \tilde{\eta}_s(dx) (\mathbb{E}_{\hat{\rho}(x)} [f(r_s); R_t > 0; t < T_0] + \mathbb{E}_{\hat{\rho}(x)} [f(-\sigma r_s); R_t < 0; t < T_0]). \end{aligned}$$

Provided that the process is started away from zero and up until the first hitting time of zero (i.e. on a single excursion) the probability of being positive is given by an independent jump process with two states under a time change. This can be seen directly from the Lamperti-Kiu representation. Setting this time-changed Markov chain with two states, $\{1, -\sigma\}$, to be the Markov chain J_t defined on Page 34 analogously to [Ross, 2010, p. 404] and $q^+ + q^- = q$ and using $\mathbb{E}_{x,z}$ to denote the expectation where the Bessel process r is started at x and the

independent jump process J_t is started at z

$$\begin{aligned}
& \int \int \tilde{\eta}_t(dx) f(y) p_s^0(x, dy) \\
&= \int_{-\infty}^{\infty} \tilde{\eta}_t(dx) \left(\mathbb{E}_{\hat{\rho}(x)} [f(r_s); J_{H_s} = 1; s < T_0] + \mathbb{E}_{\hat{\rho}(x), \frac{x}{\sigma}} [f(-\sigma r_s); J_{H_s} = -\sigma; s < T_0] \right) \\
&= \int_0^{\infty} \tilde{\eta}_t(dx) \left(\mathbb{E}_{x,1}^0 [f(r_s); J_{H_s} = 1] + \mathbb{E}_{x,1}^0 [f(-\sigma r_s); J_{H_s} = -\sigma] \right) \\
&+ \int_{-\infty}^0 \tilde{\eta}_t(dx) \left(\mathbb{E}_{-\frac{x}{\sigma}, -\sigma}^0 [f(r_s); J_{H_s} = 1] + \mathbb{E}_{-\frac{x}{\sigma}, -\sigma}^0 [f(-\sigma r_s); J_{H_s} = -\sigma] \right) \\
&= \int_0^{\infty} \pi^- n_t(dx) \left(\mathbb{E}_x^0 [f(r_s) (\pi^+ e^{-qH_s} + \pi^-)] + \mathbb{E}_x^0 [f(-\sigma r_s) \pi^+ (1 - e^{-qH_s})] \right) \\
&+ \int_{-\infty}^0 \pi^+ \bar{n}_t^\sigma(dx) \left(\mathbb{E}_{-\frac{x}{\sigma}}^0 [f(r_s) \pi^- (1 - e^{-qH_s})] + \mathbb{E}_{-\frac{x}{\sigma}}^0 [f(-\sigma r_s) (\pi^- e^{-qH_s} + \pi^+)] \right) \\
&= \int_0^{\infty} \pi^- n_t(dx) \left(\mathbb{E}_x^0 [f(r_s) (\pi^+ e^{-qH_s} + \pi^-)] + \mathbb{E}_x^0 [f(-\sigma r_s) \pi^+ (1 - e^{-qH_s})] \right) \\
&+ \int_0^{\infty} \pi^+ n_t(dx) \left(\mathbb{E}_x^0 [f(r_s) \pi^- (1 - e^{-qH_s})] + \mathbb{E}_x^0 [f(-\sigma r_s) (\pi^- e^{-qH_s} + \pi^+)] \right),
\end{aligned}$$

where we have used the representation of $\tilde{\eta}$ in terms of the Bessel entrance law n given in the statement of the Theorem and the change of variables $-\frac{x}{\sigma} \rightarrow x$ on the negative half-line. Finally, summing these together gives

$$\int \int \tilde{\eta}_t(dx) f(y) p_s^0(x, dy) = \int_0^{\infty} \pi^- n_t(dx) \mathbb{E}_x [f(r_s)] + \int_{-\infty}^0 \pi^+ \bar{n}_t^\sigma(dx) \mathbb{E}_{-\frac{x}{\sigma}} [f(r_s)]$$

and since n_t is a unique Bessel process entrance law on \mathbb{R}^+ that satisfies (2.21), see Rivero [2007], this is equivalent to

$$\begin{aligned}
\int \int \tilde{\eta}_t(dx) f(y) p_s^0(x, dy) &= \int_0^{\infty} \pi^- n_{t+s}(dx) f(x) + \int_0^{\infty} \pi^+ \bar{n}_t^\sigma(dx) f(x) \\
&= \int f(x) \tilde{\eta}(dx)
\end{aligned}$$

and thus, we have our result. □

Lemma 22 (Uniqueness of the entrance law). *Let \hat{R} be a minimal Feller process on \mathbb{R} that has a non-zero probability of jumping and can be expressed as the Lamperti-Kiu representation given in (2.13). The entrance law for the recurrent extension R of \hat{R} such that $\hat{\rho}(R_t)$ is a Bessel process is unique.*

Proof. Referring to [Blumenthal, 1992, p. 157 Section V Thm 4.2 and Section V Thm 4.6],

If $\frac{V^\lambda f(0)}{V^1 \mathbf{1}(0)}$ exists for all continuous functions f with compact support and vanishing near 0, then there exists at most one recurrent extension with continuous entrance and no sojourn at 0.

Here, V^λ is the λ -potential of the killed process and no sojourn at 0 means $\mathbb{E}_x \left[\int_0^\infty \mathbf{1}_{[X_t \in \{0\}]} dt \right] = 0$.

Since $\hat{\rho}(R_t)$ is a Bessel process that leaves zero continuously with no sojourn (as discussed in Theorem 12), the existence and well-definedness of the expression $\frac{V^\lambda f(x)}{V^1 \mathbf{1}(x)}$ for all continuous f with compact support vanishing near 0 and x in the state space guarantees the existence of at most one entrance law. Thus, in this proof we show that this expression is well-defined.

For $x \in \mathbb{R} \setminus \{0\}$ the well-definedness of the expression follows from the properties of the minimal process. This is because the potential of the minimal process is defined on all $\mathbb{R} \setminus \{0\}$. However, we are required to determine the process at zero, in particular, to show that the limits from above and below are equal.

Initially, we note that the expression is given by

$$\frac{V^\lambda f(x)}{V^1 \mathbf{1}(x)} = \frac{V^1 f(x)}{\mathbb{E}_x [1 - e^{-T_0}]}$$

where T_0 is the first hitting time of zero by \hat{R} , or equivalently R . By Theorem 12, we note that $\hat{\rho}(R_t) = r_t$, a Bessel process of index ν initiated at $\hat{\rho}(x)$. Therefore,

$$\mathbb{E}_x [1 - e^{-T_0}] = \mathbb{E}_{\hat{\rho}(x)} [1 - e^{-T_0^r}]$$

where T_0^r is the first time that the Bessel process hits zero. We would like to show the existence of the limit

$$\lim_{x \rightarrow 0} \frac{\mathbb{E}_x [1 - e^{-T_0}]}{\hat{\rho}(x)^{-2\nu}} = \lim_{x \rightarrow 0} \frac{\mathbb{E}_{\hat{\rho}(x)} [1 - e^{-T_0^r}]}{\hat{\rho}(x)^{-2\nu}}.$$

To see this, we first recall that the eigenfunctions of a Bessel process are well known and given by

$$\mathbb{E}_x [e^{-\alpha T_y^r}] = \begin{cases} \frac{x^{-\nu} I_\nu(x\sqrt{2\alpha})}{y^{-\nu} I_\nu(y\sqrt{2\alpha})} & \text{if } x < y \\ \frac{x^{-\nu} K_\nu(x\sqrt{2\alpha})}{y^{-\nu} K_\nu(y\sqrt{2\alpha})} & \text{if } x \geq y \end{cases} \quad (2.22)$$

and the asymptotic at zero is given in [Borodin and Salminen, 2002, p. 638 Appendix 2] by $K_\nu(y) \sim \frac{\Gamma(|\nu|)}{2} \left(\frac{y}{2}\right)^{-|\nu|}$. This yields

$$\mathbb{E}_x [1 - e^{\lambda T_0^r}] = 1 - \mathbb{E}_x [e^{\lambda T_0^r}] = 1 - \frac{x^{-\nu} K_\nu(x\sqrt{2\lambda})}{\frac{\Gamma(-\nu)}{2} \left(\frac{\sqrt{2\lambda}}{2}\right)^\nu}. \quad (2.23)$$

We are concerned with the limit as x approaches zero so we look for an expansion around 0 of K_ν . Employing $K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)} [I_{-\nu}(z) + I_\nu(z)]$, as given in Appendix A.1, and the series

expansions of each of the increasing modified Bessel functions

$$\begin{aligned} \frac{2 \sin(\nu\pi)}{\pi} z^{-\nu} K_{\nu}(z) &= \sum_{k=0}^{\infty} \frac{z^{2k-2\nu}}{k! \Gamma(-\nu + k + 1) 2^{2k-\nu}} - \sum_{k=0}^{\infty} \frac{z^{2k}}{k! \Gamma(\nu + k + 1) 2^{2k+\nu}} \\ &= \frac{z^{-2\nu}}{\Gamma(-\nu + 1) 2^{-\nu}} - \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu + 1) 2^{\nu}} + o(z^2). \end{aligned}$$

We can obtain an expansion around zero using Euler's reflection formula: $\Gamma(1-\nu)\Gamma(\nu) = \frac{\pi}{\sin(\nu\pi)}$ and substituting this expansion of K_{ν} into (2.23)

$$\mathbb{E}_x \left[1 - e^{\lambda T_0^r} \right] = \left(\frac{x}{2} \right)^{-2\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)}.$$

Thus, we have the existence of the limit

$$\lim_{x \rightarrow 0} \frac{\mathbb{E}_x [1 - e^{-T_0}]}{\hat{\rho}(x)^{-2\nu}} = \lim_{x \rightarrow 0} \frac{\mathbb{E}_{\hat{\rho}(x)} [1 - e^{-T_0^r}]}{\hat{\rho}(x)^{-2\nu}}.$$

Hence, using our expression for ρ in Theorem 12 under a certain time-scaling and without loss of generality, the result reduces to proving the equivalence of the limits

$$\lim_{x \downarrow 0} \frac{V^{\lambda} f(x)}{x^{-2\nu}} = \lim_{x \uparrow 0} \frac{V^{\lambda} f(x)}{\left(\frac{x}{\sigma} \right)^{-2\nu}}.$$

Before proving this equivalence of limits we first note that

$$\lim_{x \rightarrow 0} \int_0^{\infty} e^{-\lambda t} \int_0^{\infty} f(y) I_{\sqrt{\nu^2+q}} \left(\frac{xy}{t} \right) \frac{y^{\nu+1}}{x^{-\nu} t} e^{-\frac{x^2+y^2}{2t}} dy dt = 0$$

for any f Borel with compact support, which is thus bounded by a constant say K . To show this, we show the limit of the inner integral using the substitution $y = \frac{\sqrt{z}t}{x}$ and the Tauberian Theorem in [Bertoin, 1998, Chapter 0 Section 7]

$$\lim_{x \rightarrow 0} \int_0^{\infty} f(y) I_{\sqrt{\nu^2+q}} \left(\frac{xy}{t} \right) \frac{y^{\nu+1}}{x^{-\nu} t} e^{-\frac{x^2+y^2}{2t}} dy \quad (2.24)$$

$$\leq \lim_{x \rightarrow 0} K \frac{t^{\nu+1}}{x^2} e^{-\frac{x^2}{2t}} \int_0^{\infty} I_{\sqrt{\nu^2+q}}(\sqrt{z}) z^{\frac{\nu}{2}} e^{-\frac{zt}{2x^2}} dy \quad (2.25)$$

$$= \lim_{\mu \rightarrow \infty} K t^{\nu} \mu \int_0^{\infty} I_{\sqrt{\nu^2+q}}(\sqrt{z}) z^{\frac{\nu}{2}} e^{-\mu z} dy, \quad (2.26)$$

where we have used $\mu = \frac{t}{2x^2}$. Since, $I_{\sqrt{\nu^2+q}}(\sqrt{z}) z^{\frac{\nu}{2}}$ is a regularly varying function by the asymptotics of $I_{\sqrt{\nu^2+q}}$ given in [Borodin and Salminen, 2002, p. 638 Appendix 2], this is equivalent, up to a constant, to the limit

$$\lim_{z \rightarrow 0} I_{\sqrt{\nu^2+q}}(\sqrt{z}) z^{\frac{\nu}{2}} = 0.$$

This limit is also given by the asymptotics of the modified Bessel function.

We are now in a position to prove the equivalence of the limits. Considering the case as x tends to zero from above, and using the expression of the killed semigroup density in Theorem 16

$$\begin{aligned} \lim_{x \downarrow 0} \frac{V^\lambda f(x)}{x^{-2\nu}} &= \lim_{x \downarrow 0} \int_0^\infty e^{-\lambda t} \int f(y) \frac{p_t^0(x, y)}{x^{-2\nu}} dy dt \\ &= \pi^- \lim_{x \downarrow 0} \int_0^\infty e^{-\lambda t} \int_0^\infty f(y) I_{-\nu} \left(\frac{xy}{t} \right) \frac{y^{\nu+1} x^\nu}{t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} dy \\ &\quad + \frac{\pi^+}{\sigma^{\nu+2}} \lim_{x \downarrow 0} \int_0^\infty e^{-\lambda t} \int_{-\infty}^0 f(y) I_{-\nu} \left(\frac{-xy}{\sigma t} \right) \frac{|y|^{\nu+1} x^\nu}{t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} dy dt. \end{aligned}$$

Similarly, in the negative case

$$\begin{aligned} \lim_{x \uparrow 0} \frac{V^\lambda f(x)}{\left(\frac{x}{\sigma}\right)^{-2\nu}} &= \lim_{x \uparrow 0} \frac{V^1 f(x\sigma)}{(x)^{-2\nu}} \\ &= \lim_{x \uparrow 0} \int_0^\infty e^{-\lambda t} \int f(y) \frac{p_t^0(x\sigma, y)}{(x)^{-2\nu}} dy dt \\ &= \pi^- \lim_{x \uparrow 0} \int_0^\infty e^{-\lambda t} \int_0^\infty f(y) I_{-\nu} \left(\frac{xy}{t} \right) \frac{y^{\nu+1} x^\nu}{t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} dy \\ &\quad + \frac{\pi^+}{\sigma^{\nu+2}} \lim_{x \uparrow 0} \int_0^\infty e^{-\lambda t} \int_{-\infty}^0 f(y) I_{-\nu} \left(\frac{-xy}{\sigma t} \right) \frac{|y|^{\nu+1} x^\nu}{t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} dy dt. \end{aligned}$$

Thus, if we can show the existence of the two limits in the last equality, then we are done. We prove the positive case and the negative case follows similarly.

By Fubini, the limit can be expressed in terms of the resolvent density of a Bessel process, which is given in [Borodin and Salminen, 2002, p. 133 Appendix 1 Section 21]

$$\begin{aligned} &\int_0^\infty e^{-\lambda t} \int_0^\infty f(y) I_{-\nu} \left(\frac{xy}{t} \right) \frac{y^{\nu+1} x^\nu}{t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} dy dt \\ &= \int_0^\infty f(y) y^{2\nu} \int_0^\infty I_{-\nu} \left(\frac{xy}{t} \right) \frac{y^{-\nu+1}}{x^{-\nu} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2t}} dt dy \\ &= \int_0^x f(y) y^{-2\mu} x^{-\mu} K_\mu(x\sqrt{2\lambda}) y^{-\mu} I_\mu(y\sqrt{2\lambda}) y^{2\mu+1} dy \\ &\quad + \int_x^\infty f(y) y^{-2\mu} y^{-\mu} K_\mu(y\sqrt{2\lambda}) x^{-\mu} I_\mu(x\sqrt{2\lambda}) y^{2\mu+1} dy \end{aligned}$$

where we have let $\mu = -\nu$ and split the integral. Dealing with the lower integral limit first, by l'Hôpital's rule

$$\begin{aligned} &\lim_{x \rightarrow 0} x^{-\mu} K_\mu(x\sqrt{2\lambda}) \int_0^x f(y) y^{-2\mu} y^{-\mu} I_\mu(y\sqrt{2\lambda}) y^{2\mu+1} dy \\ &= \lim_{x \rightarrow 0} \frac{f(x) x^{1-\mu} I_\mu(x\sqrt{2\lambda}) x^{-\mu} K_\mu(x\sqrt{2\lambda})^2}{K_{\mu+1}(x\sqrt{2\lambda})} \\ &\sim K f(x) x^{2-2\mu} \rightarrow 0, \end{aligned}$$

where we have used the asymptotics of the Bessel functions, see [Borodin and Salminen, 2002, p. 638 Appendix 2]. Finally, for the upper integral

$$\begin{aligned} & \lim_{x \rightarrow 0} x^{-\mu} I_{\mu}(x\sqrt{2\lambda}) \int_x^{\infty} f(y) y^{-2\mu} y^{-\mu} K_{\mu}(y\sqrt{2\lambda}) y^{2\mu+1} dy \\ & \sim \frac{(\sqrt{2\lambda})^{\mu}}{\Gamma(\mu+1)2^{\mu}} \int_0^{\infty} f(y) y^{-2\mu} y^{-\mu} K_{\mu}(y\sqrt{2\lambda}) y^{2\mu+1} dy. \end{aligned}$$

By the finiteness of the integral for a Borel f with compact support, we have the existence of the limit. □

Now that we have the existence and uniqueness of the entrance law of the process, we are in a position to calculate the semigroup density explicitly.

Lemma 23. *Let \hat{R} be a stochastic process on \mathbb{R} with a Lamperti-Kiu representation of the form (2.13) that is killed at the first time it hits zero. The recurrent extension of this minimal process has a semigroup density that is given by (2.20) in Theorem 17.*

Proof. In this proof we will frequently use the expression for resolvents of Markov extensions given in Rogers [1983] for processes that leave zero continuously by

$$U^{\lambda} f(x) = V_0^{\lambda} f(x) + \mathbb{E}_x \left[e^{-\lambda T_0} \right] \frac{\tilde{\eta}_{\lambda}(f)}{\lambda \tilde{\eta}_{\lambda}(1)}, \quad (2.27)$$

for a compact, Borel f . Here U^{λ} , V_0^{λ} are the resolvents of the extension and the minimal process killed at zero respectively, T_0 is the first hitting time of zero and $\tilde{\eta}_{\lambda}$ is the resolvent of the entrance law.

To begin with, we note that this equation is satisfied by the entrance law, $n(dx)$, of the unique Bessel process extension for f and $f(-\sigma \cdot)$:

$$\begin{aligned} U_r^{\lambda} f(x) &= V_{0,r}^{\lambda} f(x) + \mathbb{E}_x \left[e^{-\lambda T_0^r} \right] \frac{n_{\lambda}(f)}{\lambda n_{\lambda}(1)} \\ U_r^{\lambda} f(-\sigma x) &= V_0^{\lambda} f(-\sigma x) + \mathbb{E}_x \left[e^{-\lambda T_0^r} \right] \frac{n_{\lambda}(f(-\sigma \cdot))}{\lambda n_{\lambda}(1)} \end{aligned}$$

and therefore,

$$\frac{\mathbb{E}_x \left[e^{-\lambda T_0^r} \right]}{\lambda n_{\lambda}(1)} = \frac{U_r^{\lambda} f(x) - V_{0,r}^{\lambda} f(x)}{n_{\lambda}(f)} = \frac{U_r^{\lambda} f(-\sigma x) - V_0^{\lambda} f(-\sigma x)}{n_{\lambda}(f(-\sigma \cdot))}.$$

Returning to the process R in the statement of the theorem, we have already shown (see Lemma 21) that the entrance law can be expressed

$$\tilde{\eta}_{\lambda}(f) = \pi^{-} n_{\lambda}(f) + \pi^{+} n_{\lambda}(f(-\sigma \cdot)) \quad (2.28)$$

and we also note that since $\pi^{+} + \pi^{-} = 1$ we have $\lambda \tilde{\eta}_{\lambda}(1) = \lambda n_{\lambda}(1)$.

Therefore, recalling the resolvent equation of R given by (2.27) and using this expression of the entrance law (2.28),

$$U^\lambda f(x) = V_0^\lambda f(x) + \frac{\mathbb{E}_x [e^{-\lambda T_0}]}{\lambda \tilde{\eta}_\lambda(1)} (\pi^- n_\lambda(f) + \pi^+ n_\lambda(f(-\sigma \cdot)))$$

and since $\lambda \tilde{\eta}_\lambda(1) = \lambda n_\lambda(1)$, for the law with respect to the Bessel process n_λ and the law with respect to R $\tilde{\eta}_\lambda$, and $\mathbb{E}_x [e^{-\lambda T_0}] = \mathbb{E}_x [e^{-\lambda T_0^r}]$ for $x > 0$ we can substitute (2.27)

$$\begin{aligned} U^\lambda f(x) &= V_0^\lambda f(x) + \frac{U_r^\lambda f(x) - V_{0,r}^\lambda f(x)}{n_\lambda(f)} \pi^- n_\lambda(f) + \frac{U_r^\lambda f(-\sigma x) - V_0^\lambda f(-\sigma x)}{n_\lambda(f(-\sigma \cdot))} \pi^+ n_\lambda(f(-\sigma \cdot)) \\ &= V_0^\lambda f(x) + \left(U_r^\lambda f(x) - V_{0,r}^\lambda f(x) \right) \pi^- + \left(U_r^\lambda f(-\sigma x) - V_0^\lambda f(-\sigma x) \right) \pi^+. \end{aligned}$$

By the uniqueness of the Laplace transform we now have

$$p_t(x, y) = \begin{cases} p_t^0(x, y) + \pi^- q_t(x, y) - \pi^- q_t^0(x, y) & y \geq 0 \\ p_t^0(x, y) + \frac{\pi^+}{\sigma} q_t\left(x, -\frac{y}{\sigma}\right) - \frac{\pi^+}{\sigma} q_t^0\left(x, -\frac{y}{\sigma}\right) & y < 0, \end{cases}$$

where $p_t^0(x, y)$ is the semigroup density of the minimal process \hat{R} killed at the first time it hits zero and $q_t(x, y), q_t^0(x, y)$ are the semigroup densities of the unique Bessel extension and the minimal Bessel process killed at zero respectively. Similarly, we can use the same methodology when $x < 0$ to obtain

$$p_t(x, y) = \begin{cases} p_t^0(x, y) + \pi^- q_t\left(-\frac{x}{\sigma}, y\right) - \pi^- q_t^0\left(-\frac{x}{\sigma}, y\right) & y \geq 0 \\ p_t^0(x, y) + \frac{\pi^+}{\sigma} q_t\left(-\frac{x}{\sigma}, -\frac{y}{\sigma}\right) - \frac{\pi^+}{\sigma} q_t^0\left(-\frac{x}{\sigma}, -\frac{y}{\sigma}\right) & y < 0. \end{cases}$$

which gives the result in the statement of the Lemma.

The probability of the process being positive as x tends to zero follows because

$$\lim_{x \rightarrow 0} \int_0^\infty I_{\sqrt{\nu+2q}}\left(\frac{xy}{t}\right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} dy = 0$$

as given in (2.24), and therefore

$$\lim_{x \rightarrow 0} \mathbb{P}_x(R_t > 0) = \lim_{x \rightarrow 0} \pi^- \int_0^\infty I_\nu\left(\frac{xy}{t}\right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} dy = \pi^-$$

by the conservative property of the Bessel process. □

Proof of Theorem 17 Part I: The case of almost surely continuous paths

In the case of continuous paths, we have already shown, see Theorem 16, that, when the process is killed at the first time it hits zero, the state space is reduced to \mathbb{R}^+ . The characterisation of

this case was shown in Theorem 10. Thus, we only need consider recurrent extensions of the Bessel process. For this we construct a recurrent extension of the form in Lejay [2006].

Rivero [2005] has shown that the Bessel process has a unique recurrent extension on \mathbb{R}^+ , and since our process can enter the state space in the positive or negative direction we are limited to leaving zero via two possible ways, one positive Bessel process path and one negative Bessel process path, with possibly different time-scalings. Up to a time-scaling, the result $\hat{\rho}(R_t) = r_t$ in Theorem 12 means that these two extensions are given by r_t and $-\sigma r_t$. Both these processes hit zero at the same time so let the closure of the zeros of these processes be given by $\mathcal{Z} = \overline{\{t \geq 0 : r_t = 0\}}$, see Lejay [2006] and the set of references therein. By Lejay [2006], the set $\mathbb{R}^+ \setminus \mathcal{Z}$ can be decomposed as a countable union of excursion intervals J_n . On each of these intervals we associate a random variable ϵ_n that chooses the positive or negative excursion. Thus, the process R can be expressed

$$R_t = \epsilon_n r_t + (1 - \epsilon_n)(-\sigma r_t)$$

where ϵ_n takes the value 0 or 1 in each interval J_n . By the Markov and time-homogeneous properties of the process, the future distribution of R_t must have no memory of previous excursions and no time dependence so the ϵ_n that choose the sign of the excursion are independent and identically distributed. This means that ϵ_n must have a Bernoulli distribution. In the following, we use a Bernoulli distribution with parameter β . This can be seen because any other distribution for choosing between the two extensions would require another variable; voiding the time homogeneous and Markov properties. Thus, we are limited to the skew Bessel processes, with a possible changing in scale at zero, and their h -transforms.

Following the methodology of Lemma 21, we know that the entrance law of the skew Bessel process is given by

$$\tilde{\eta}_t(dx) = \mathbf{1}_{[x>0]} \beta n_t(dx) + \mathbf{1}_{[x<0]} (1 - \beta) \bar{n}_t^\sigma(dx), \quad (2.29)$$

where $n_t(dx)$ is the unique entrance law of the Bessel process with the same index $\nu \in (-1, 0)$ on \mathbb{R}^+ and $\beta \in [0, 1]$ is the probability that the process is positive given that it was started at zero, which is a constant by self-similarity.

Since Rivero [2007] has shown that the entrance law of the Bessel process is unique and satisfies the entrance law equation given in [Blumenthal, 1992, p. 137]

$$\int \int f(y) p_s^0(x, dy) n_t(dx) = \int f(y) n_{t+s}(dy) \quad (2.30)$$

for $x > 0$, then the entrance law (2.29) also satisfies this equation. However, because the process is only defined on a half-line (\mathbb{R}^+ or \mathbb{R}^-) up until its first stopping time the expression $\frac{V^\lambda f(x)}{V^\lambda \mathbf{1}(x)}$ need only be well-defined on this half-line, see [Blumenthal, 1992, p. 157 Section V Thm 4.2]. We do not require existence and uniqueness of the expression at the origin. This means that we do not have the uniqueness property we had in the jump case in Lemma 22 and so we are

free to choose any $\beta \in [0, 1]$ since any β in this range satisfies (2.30).

We can now follow a similar methodology to Lemma 23. For an $x > 0$, the resolvent of the semigroup density, $U^\lambda f(x)$ can be defined as Rogers [1983] and is given in (2.27). In this equation, V^λ represents the resolvent of the process killed at zero, which is equivalent to the resolvent of a Bessel process killed at zero in our case since we are restricted to \mathbb{R}^+ . Furthermore, T_0 is equivalent to the stopping time of a Bessel process; $\tilde{\eta}_\lambda(1) = n_\lambda(1)$ and $\tilde{\eta}_\lambda(f)$ is given by (2.29). This yields

$$U^\lambda f(x) = V^\lambda f(x) + \frac{\mathbb{E}_x [e^{-\lambda T_0}]}{\lambda \tilde{\eta}_\lambda(1)} \beta n_\lambda(f) + \frac{\mathbb{E}_x [e^{-\lambda T_0}]}{\lambda \eta_\lambda(1)} (1 - \beta) n_\lambda(f(-\sigma \cdot))$$

and since the Bessel process itself also satisfies (2.27), we have

$$U^\lambda f(x) = V_r^\lambda f(x) + (U_r^\lambda f(x) - V_r^\lambda f(x)) \beta n_\lambda(f) + (U_r^\lambda f(-\sigma x) - V_r^\lambda f(-\sigma x)) (1 - \beta) n_\lambda(f(-\sigma \cdot)).$$

Therefore, the resolvent equation for the semigroup density we require is given by

$$\begin{aligned} U^\lambda f(x) &= \int_0^\infty e^{-\lambda t} f(y) (q_t^0(x, y) + \beta(q_t(x, y) - q_t^0(x, y))) dy \\ &\quad + (1 - \beta) \int_{-\infty}^0 e^{-\lambda t} f(y) \left(\frac{q_t(x, \frac{y}{\sigma})}{\sigma} - \frac{q_t(x, \frac{y}{\sigma})}{\sigma} \right) dy. \end{aligned}$$

Utilising the same technique when $x < 0$ leaves

$$p_t(x, y) = \begin{cases} [\beta I_\nu(\frac{xy}{t}) + (1 - \beta) I_{-\nu}(\frac{xy}{t})] \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} & \text{if } x \geq 0, y \geq 0 \\ \beta (I_{-\nu}(-\frac{xy}{\sigma t}) - I_\nu(-\frac{xy}{\sigma t})) \frac{|y|^{\nu+1}}{x^\nu \sigma^{\nu+2} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x \geq 0, y < 0 \\ (1 - \beta) (I_{-\nu}(-\frac{xy}{\sigma t}) - I_\nu(-\frac{xy}{\sigma t})) \frac{y^{\nu+1} \sigma^\nu}{|x|^\nu t} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2t}} & \text{if } x < 0, y \geq 0 \\ [(1 - \beta) I_\nu(\frac{xy}{\sigma^2 t}) + \beta I_{-\nu}(\frac{xy}{\sigma^2 t})] \frac{|y|^{\nu+1}}{|x|^\nu \sigma^2 t} e^{-\frac{x^2+y^2}{2\sigma^2 t}} & \text{if } x < 0, y < 0 \end{cases}.$$

Remark 24. In the case when $\sigma = 1$, we obtain the semigroup density found in Watanabe.

Since the skew Bessel process with a variable Gaussian coefficient has a semigroup density which takes the form (2.2), it also has the time inversion property by the result in Lawi [2008]. Moreover, it is another example of a process whose function Φ is asymmetric.

We now show that $\mathbb{P}_0(R_t > 0) = \beta$, where β is defined as above. To show this we employ the expression for the asymptotic value at zero of the modified Bessel function given in [Borodin and Salminen, 2002, Appendix 2] by

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2} \right)^\nu. \quad (2.31)$$

We would first like to show

$$\lim_{x \rightarrow 0} \int_0^\infty I_{-\nu} \left(\frac{xy}{t} \right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} dy = 0.$$

Using a similar methodology to (2.24),

$$\begin{aligned} & \lim_{x \rightarrow 0} \int_0^\infty I_{-\nu} \left(\frac{xy}{t} \right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} dy \\ & \leq \lim_{x \rightarrow 0} \frac{t^\nu}{x^{2\nu+2}} e^{-\frac{x^2}{2t}} \int_0^\infty I_{-\nu}(\sqrt{z}) z^{\frac{\nu}{2}} e^{-\frac{zt}{2x^2}} dz \\ & = \lim_{\mu \rightarrow \infty} C_1 \times \mu^{\nu+1} \int_0^\infty I_{-\nu}(\sqrt{z}) z^{\frac{\nu}{2}} e^{-\mu z} dz. \end{aligned}$$

for a C_1 that depends only on t and is independent of x and y . Employing (2.31), we can see that $I_{-\nu}(\sqrt{z}) z^{\frac{\nu}{2}}$ is a slowly varying function

$$\frac{I_{-\nu}(\sqrt{cz}) (cz)^{\frac{\nu}{2}}}{I_{-\nu}(\sqrt{z}) z^{\frac{\nu}{2}}} \sim c^{\frac{-\nu}{2}} c^{\frac{\nu}{2}} \sim 1$$

and therefore, by the Tauberian Theorem in [Bertoin, 1998, Chapter 0 Section 7]

$$\lim_{\mu \rightarrow \infty} \mu \int_0^\infty I_{-\nu}(\sqrt{z}) z^{\frac{\nu}{2}} e^{-\mu z} dz = C_2$$

for a C_2 independent of μ . Thus,

$$\lim_{\mu \rightarrow \infty} C_1 \times \mu^{\nu+1} \int_0^\infty I_{-\nu}(\sqrt{z}) z^{\frac{\nu}{2}} e^{-\mu z} dz = 0.$$

This means that, by the definition of a probability measure

$$\lim_{x \rightarrow 0} \int_0^\infty I_{\nu} \left(\frac{xy}{t} \right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} dy = 1$$

and

$$\lim_{x \rightarrow 0} \int_0^\infty I_{-\nu} \left(\frac{xy}{t} \right) \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} dy = 0$$

Thus, from the form of the semigroup density for the skew Bessel process (2.34), we have

$$\lim_{x \rightarrow 0} \mathbb{P}_x(R_t > 0) = \beta \quad \text{and} \quad \lim_{x \rightarrow 0} \mathbb{P}_x(R_t < 0) = 1 - \beta.$$

The proof of the Theorem follows from a combination of the result in the case of continuous paths, given by Section 2.4.3, and the result in the case of processes with jumps, given by Lemma 23.

These results fully characterise, up to an h -transform, all Markov processes on \mathbb{R} satisfying (H1-3) and enjoying the Markov property as one of the following:-

- Brownian motion;
- Skew Bessel process (possibly stopped at zero or a possible changing of time-scaling at zero);
- Generalised Dunkl Process (possibly stopped at zero or a possible changing of time-scaling at zero and with a possible changing of jump parameter at zero).

Alternate Proof of Theorem 17 Part I: The case of almost surely continuous paths

As a final part of this chapter, we calculate the semigroup density of the process in the case of continuous paths by extending the methods in Blei [2012] and using a PDE approach.

In the case of continuous paths, we have already shown that the process must be a skew Bessel process, with a possible change in time-scaling at the origin, which has a Bernoulli choice of choosing a positive or negative path at zero.

For this reason, our aim is to provide the semigroup density for the skew Bessel process with a possible change in time-scaling at zero. This is equivalent to the diffusion on \mathbb{R} whose infinitesimal generator applied to an $f \in \mathcal{D}_{\mathcal{A}_R}$ is given by

$$\begin{aligned}\mathcal{A}_R f(x) &= \sigma(\text{sgn}(x))^2 \frac{2\nu + 1}{2x} f'(x) + \frac{\sigma(\text{sgn}(x))^2}{2} f''(x) \\ \mathcal{A}_R f(0) &= \lim_{h \rightarrow 0} \mathcal{A}_R f(h) = 0\end{aligned}$$

namely, the skew Bessel process whose time-scaling potentially changes depending on whether the process is on the positive or negative half-line. We note that we provide some restrictions on $\mathcal{D}_{\mathcal{A}_R}$ in the following. Apart from being interesting in itself, the semigroup density is enough to show the time inversion property by the result in Lawi [2008]. We first determine the diffusion through its scale function and speed measure.

Since the process is a Bessel process up until its first hitting time of zero, we note that it must be a constant multiple of the scale function of the Bessel process on each of these intervals. For this reason, we use the scale function

$$S(x) = \begin{cases} \frac{1}{\beta|2\nu|} x^{-2\nu} & \text{if } x > 0 \\ \frac{-1}{|2\nu|(1-\beta)} \left(\frac{|x|}{\sigma}\right)^{-2\nu} & \text{if } x \leq 0, \end{cases}$$

for $\beta \in [0, 1]$ and consequently, the speed measure is inferred to be

$$m(dx) = \begin{cases} 2\beta(x^{2\nu+1})dx & \text{if } x > 0 \\ 2(1-\beta)\frac{|x|^{2\nu+1}}{\sigma^{2\nu+2}}dx & \text{if } x \leq 0. \end{cases}$$

We note that this parametrisation of the scale function makes intuitive sense because of Theorem 12. We could choose different constants and get a different expression for the same process with different parameters. Since the scale function is increasing and continuous on \mathbb{R} ; the speed

measure is a Borel measure and we assume the killing measure is zero these two functions define a unique diffusion on \mathbb{R} .

In the following we refer to [Revuz and Yor, 2005, Chapter VIII] and, in particular [Revuz and Yor, 2005, Chapter VIII, Thm VIII.3.2], for the construction of the generator using scale functions and the speed measure. For the domain, we firstly require that the function be continuously differentiable with respect to the scale function. For a C^2 function f , this derivative is given by

$$D_S f(x) = \begin{cases} f'(x)x^{2\nu+1}\beta & \text{if } x > 0 \\ -f'(x)|x|^{2\nu+1}\sigma^{-2\nu}(\beta - 1) & \text{if } x \leq 0, \end{cases}$$

and so, in order to maintain this continuity, we require that in the domain of the generator we have the equality

$$\lim_{x \downarrow 0} \beta f'(x)x^{2\nu+1} = \lim_{x \uparrow 0} (1 - \beta)\sigma^{-2\nu} f'(x)|x|^{2\nu+1}. \quad (2.32)$$

We would also like continuity of the generator itself, but once we have the above condition this is simply $D_M D_S = \mathcal{A}_R$. This continuity condition is held by any solution to the eigenvalue problem associated with the Bessel process provided that the constant associated with the eigenvalue problem is taken to be divided by σ^2 . For example, if ψ is a solution to the eigenvalue problem for the Bessel process on \mathbb{R}^- then

$$\mathcal{A}_r \psi(x\sqrt{2\lambda}) = \lambda \psi(x\sigma^{-1}\sqrt{2\lambda})$$

and letting $\lambda = \alpha\sigma^{-2}$

$$\sigma^2 \mathcal{A}_r \psi(x\sigma^{-1}\sqrt{2\alpha}) = \alpha \psi(x\sigma^{-1}\sqrt{2\alpha}),$$

which is the solution to the eigenvalue problem in the case of R on the positive line.

We would like to use this construction of the process to find the semigroup density via the Green function method. In order to do this we first define the following stopping times

$$\begin{aligned} T_y^r &:= \inf\{t \geq 0; r_t = y\} \\ T_y &:= \inf\{t \geq 0; R_t = y\}. \end{aligned}$$

Then, since the eigenfunctions of the positive Bessel process are well known, by the properties of a Bessel process

$$\mathbb{E}_x [e^{-\alpha T_y^r}] = \begin{cases} \frac{x^{-\nu} I_\nu(x\sqrt{2\alpha})}{y^{-\nu} I_\nu(y\sqrt{2\alpha})} & \text{if } x < y \\ \frac{x^{-\nu} K_\nu(x\sqrt{2\alpha})}{y^{-\nu} K_\nu(y\sqrt{2\alpha})} & \text{if } x \geq y. \end{cases} \quad (2.33)$$

Therefore, if $x \geq y > 0$, then by self-similarity or equivalently by the properties of a time-change

$$\mathbb{E}_x [e^{-\alpha T_y}] = \mathbb{E}_x [e^{-\alpha T_y^r}] = \frac{(x\sqrt{2\alpha})^{-\nu} K_\nu(x\sqrt{2\alpha})}{(y\sqrt{2\alpha})^{-\nu} K_\nu(y\sqrt{2\alpha})}$$

and if $x \leq y < 0$ then

$$\mathbb{E}_x [e^{-\alpha T_y}] = \mathbb{E}_{\frac{x}{\sigma}} [e^{-\alpha T_{\frac{y}{\sigma}}^r}] = \frac{(|x|\sqrt{2\alpha})^{-\nu} K_\nu(\frac{|x|}{\sigma}\sqrt{2\alpha})}{(|y|\sqrt{2\alpha})^{-\nu} K_\nu(\frac{|y|}{\sigma}\sqrt{2\alpha})}.$$

On account of this, the possible solutions are given by

$$\psi^-(x) = \begin{cases} \frac{2^{\nu+1}}{\Gamma(-\nu)} \tilde{K}_\nu(x\sqrt{2\alpha}) & \text{if } x \geq 0 \\ A_1 \frac{2^{\nu+1}}{\Gamma(-\nu)} \tilde{K}_\nu\left(\frac{|x|}{\sigma}\sqrt{2\alpha}\right) + B_1 2^\nu \Gamma(\nu+1) \tilde{I}_\nu\left(\frac{|x|}{\sigma}\sqrt{2\alpha}\right) & \text{if } x < 0, \end{cases}$$

where $\tilde{K}_\nu(z) = z^{-\nu} K_\nu(z)$ and $\tilde{I}_\nu(z) = z^{-\nu} I_\nu(z)$. The associated increasing solution is then given by

$$\psi^+(x) = \begin{cases} A_2 \frac{2^{\nu+1}}{\Gamma(-\nu)} \tilde{K}_\nu(x\sqrt{2\alpha}) + B_2 2^\nu \Gamma(\nu+1) \tilde{I}_\nu(x\sqrt{2\alpha}) & \text{if } x \geq 0 \\ \frac{2^{\nu+1}}{\Gamma(-\nu)} \tilde{K}_\nu\left(\frac{|x|}{\sigma}\sqrt{2\alpha}\right) & \text{if } x < 0 \end{cases}$$

for constants A_1, B_1, A_2, B_2 . We have chosen the fixed constants associated with each Bessel function because

$$\lim_{x \downarrow 0} \frac{2^{\nu+1}}{\Gamma(-\nu)} \tilde{K}_\nu(x\sqrt{2\alpha}) = \lim_{x \downarrow 0} 2^\nu \Gamma(\nu+1) \tilde{I}_\nu(x\sqrt{2\alpha}) = 1.$$

By the condition (2.32) we must have the equality of limits

$$\lim_{x \downarrow 0} \beta \psi'_-(x) x^{2\nu+1} = \lim_{x \uparrow 0} (1-\beta) \sigma^{-2\nu} \psi'_-(x) |x|^{2\nu+1}$$

and similarly for the increasing function. This allows us to compute the constants.

By the above, noting that the I_ν terms disappear in the limit and that the coefficients should add to one, we have that

$$\begin{aligned} A_1 &= \frac{\beta}{\beta-1}, & A_2 &= \frac{\beta-1}{\beta}, \\ B_1 &= 1-A_1, & B_2 &= 1-A_2. \end{aligned}$$

With these solutions, and by denoting differentiation with respect to the scale function S as ψ'_S , we obtain that the Wronskian of the pair is

$$\begin{aligned} W_\alpha &= \psi'_{+S} \psi_- - \psi'_{-S} \psi_+ \\ &= 2^{\nu+1} \frac{\Gamma(\nu+1)}{\Gamma(-\nu)} \alpha^{-\nu} \end{aligned}$$

and thus, we can compute the resolvent density for each case

$$r_\alpha(x, y) = \begin{cases} \frac{(yx)^{-\nu}}{\beta} [\beta I_\nu(x\sqrt{2\alpha}) + (1-\beta)I_{-\nu}(x\sqrt{2\alpha})] K_\nu(y\sqrt{2\alpha}) & \text{if } 0 < x \leq y \\ (y|x|)^{-\nu} \sigma^\nu \left(I_\nu\left(\frac{|x|\sqrt{2\alpha}}{\sigma}\right) - I_{-\nu}\left(\frac{|x|\sqrt{2\alpha}}{\sigma}\right) \right) & \text{if } x < 0 < y \\ \frac{|xy|^{-\nu}}{\beta-1} \sigma^{2\nu} \left[(1-\beta)I_\nu\left(\frac{|x|\sqrt{2\alpha}}{\sigma}\right) + \beta I_{-\nu}\left(\frac{|x|\sqrt{2\alpha}}{\sigma}\right) \right] K_\nu\left(\frac{|y|\sqrt{2\alpha}}{\sigma}\right) & \text{if } x \leq y < 0. \end{cases}$$

Thus, by the expression for the Laplace transform [Gradshteyn and Ryzhik [2007] p. 712 6.653]

$$\int_0^\infty e^{-\frac{1}{2}t} e^{-\frac{a^2+b^2}{2t}} I_\nu\left(\frac{ab}{t}\right) \frac{1}{t} dt = 2I_\nu(a)K_\nu(b)$$

the semigroup densities are given by

$$p_t(x, y) = \begin{cases} [\beta I_\nu\left(\frac{xy}{t}\right) + (1-\beta)I_{-\nu}\left(\frac{xy}{t}\right)] \frac{y^{\nu+1}}{x^\nu t} e^{-\frac{x^2+y^2}{2t}} & \text{if } x \geq 0, y \geq 0 \\ \beta \left(I_{-\nu}\left(-\frac{xy}{\sigma t}\right) - I_\nu\left(-\frac{xy}{\sigma t}\right) \right) \frac{|y|^{\nu+1}}{x^\nu \sigma^{\nu+2} t} e^{-\frac{x^2}{2t}} e^{-\frac{y^2}{2\sigma^2 t}} & \text{if } x \geq 0, y < 0 \\ (1-\beta) \left(I_{-\nu}\left(-\frac{xy}{\sigma t}\right) - I_\nu\left(-\frac{xy}{\sigma t}\right) \right) \frac{y^{\nu+1} \sigma^\nu}{|x|^\nu t} e^{-\frac{x^2}{2\sigma^2 t}} e^{-\frac{y^2}{2t}} & \text{if } x < 0, y \geq 0 \\ [(1-\beta)I_\nu\left(\frac{xy}{\sigma^2 t}\right) + \beta I_{-\nu}\left(\frac{xy}{\sigma^2 t}\right)] \frac{|y|^{\nu+1}}{|x|^\nu \sigma^2 t} e^{-\frac{x^2+y^2}{2\sigma^2 t}} & \text{if } x < 0, y < 0 \end{cases} \quad (2.34)$$

Since the skew Bessel process with variable Gaussian coefficient has a semigroup density which takes the form (2.2) it also has the time inversion property by the result in Lawi [2008]. Moreover, it is another example of a process whose function Φ is asymmetric.

Intuition Behind the Choice of Scale Function in the Semimartingale Case

Having chosen the scale function in the above with a particular set of parameters, we give a reasoning behind this choice of parameters.

If we consider the generator of the skew Bessel process then for $\nu \in (-\frac{1}{2}, 0)$, we have a semimartingale by the work of Blei [2012] that is given by

$$R_t = R_0 + \int_0^t \frac{2\nu+1}{2R_s} \sigma(\text{sgn}(R_s))^2 ds + \int_0^t \sigma(\text{sgn}(R_s)) dB_s, \quad (2.35)$$

where $\sigma(\text{sgn}(x)) = 1$ when x is positive and $\sigma(\text{sgn}(x)) = \sigma$ when x is negative. However, this semimartingale is not defined uniquely at zero. Following the methods of Blei [2012], we can see that using the function G_σ and its inverse H_σ given by

$$G_\sigma(z) = \left(\frac{|z|}{\sigma(\text{sgn}(x))} \right)^{-2\nu} \frac{\text{sgn}(z)}{|2\nu|} \quad \text{and} \quad H_\sigma(z) = |2\nu|^{-\frac{1}{2\nu}} \text{sgn}(x) \sigma(\text{sgn}(x)) |x|^{-\frac{1}{2\nu}}$$

we have $G_\sigma(R_t) = Y_t$ and $R_t = H_\sigma(Y_t)$, where Y is as in Blei [2012]

$$Y_t = y_0 + \int_0^t \tilde{\sigma}(Y_s) dB_s + \frac{1}{2} (L_+^Y(t, 0) - L_-^Y(t, 0))$$

$$(\beta + 1)\hat{L}^Y(t, 0) = L_+^Y(t, 0) - L_-^Y(t, 0),$$

where $2\hat{L}^Y(t, 0) = L_+^Y(t, 0) + L_-^Y(t, 0)$ and we define $L_m^R(t, y)$ to be the local time of R with respect to our speed measure

$$m(dx) = \begin{cases} 2\beta(x^{2\nu+1})dx & \text{if } x > 0 \\ 2(1-\beta)\frac{|x|^{2\nu+1}}{\sigma^{2\nu+2}}dx & \text{if } x \leq 0 \end{cases}$$

so that it satisfies $L_m^R(t, y) = L_+^Y(t, y)$ and

$$\int_0^t f(R_s)ds = \int f(y)L_m^R(t, y)m(dy).$$

Thus, we have that $R_t = H_\sigma(Y_t)$ satisfies

$$\begin{cases} R_t = R_0 + \int_0^t \frac{2\nu+1}{2R_s} \sigma(\text{sgn}(R_s))^2 ds + \int_0^t \sigma(\text{sgn}(R_s)) dB_s \\ L_m^R(t, 0) - L_m^R(t, 0-) = (\beta + 1)\hat{L}_m^R(t, 0). \end{cases}$$

In Blei [2012], Y is mapped to a local martingale Z by a function

$$K(x) = \begin{cases} \frac{1-\beta}{2}x & \text{if } x \geq 0 \\ \frac{3+\beta}{2}x & \text{if } x < 0. \end{cases}$$

Composing the functions K and G_σ and dividing by the constant $((1-\beta)\beta)^{-1}$, the scale function is given by

$$S(x) = \begin{cases} \frac{1}{\beta|2\nu|}x^{-2\nu} & \text{if } x > 0 \\ \frac{1}{|2\nu|(1-\beta)}\left(\frac{|x|}{\sigma}\right)^{-2\nu} & \text{if } x \leq 0 \end{cases}$$

because this gives a local martingale when applied to R .

Chapter 3

Restrictions Provided by the Time Inversion Property on \mathbb{R}^n

3.1 Introduction

The literature concerning the time inversion property on \mathbb{R}^n for $n \geq 2$ is sparse. Unlike the time inversion property in one dimension, it was not considered by Watanabe [1975] and was only explored eventually as a generalisation in Gallardo and Yor [2005]. This is probably the reason that few examples of the time inversion property on \mathbb{R}^n have been studied. However, despite the literature being meagre, Gallardo and Yor [2005] and Lawi [2008] were able to show that, under some minor conditions on the Markov process, the semigroup density derived on \mathbb{R} also serves as a necessary and sufficient condition for the time inversion property to hold on \mathbb{R}^n . This is covered in much greater detail in Section 1.2. The semigroup density derived by Lawi, Gallardo et al. also permitted them to list several examples of processes with the time inversion property on \mathbb{R}^n , but up to this point the list was restricted to Brownian motion, generalised Dunkl processes and the Wishart processes.

In a parallel direction, Vuolle-Apiala [2012] also studied processes with the time inversion property on \mathbb{R}^n . However, Vuolle-Apiala only considered processes with almost sure continuous paths in \mathbb{R}^n (a subset of the processes considered by Gallardo and Yor [2005]) that were polar at the origin. In this restricted class, he was able to show that the restriction of rotation invariance, given by

(RI) R_t under \mathbb{P}_x has the same finite dimensional distributions as $T^{-1}(R_t)$ under $\mathbb{P}_{T(x)}$ for all rotations $T \in O(n)$,

combined with 2-self-similarity were sufficient restrictions on a diffusion in \mathbb{R}^n to mean that it was guaranteed to enjoy the time inversion property. To clarify, we note that in (RI) we take all rotations that are also a member of the set $O(n)$ - the set of orthogonal matrices on \mathbb{R}^n . This led to the possibility of several more classes of processes with the time inversion property on \mathbb{C} by taking the skew product representation, see Section 1.6, with a Bessel process as the radial part guaranteeing self-similarity.

In this chapter, we extend the work of Gallardo and Yor [2005] and Lawi [2008] to go some way to completely determining the class of all processes satisfying Lawi's restrictions that enjoy the time inversion property on \mathbb{R}^n .

This chapter is laid out as follows. In Section 3.2, we review the time inversion property on \mathbb{R}^n and recall the bijective change of coordinates to n -spherical notation which lends itself more readily to the self-similar property; a key property with strong links to time inversion. As an extension of the work in the area of continuous paths by Watanabe [1975] and Vuolle-Apiala [2012], we consider the jumps associated with processes enjoying the time inversion property. Consequently, Section 3.3 explicitly determines the jumps that are permitted by a process with the time inversion property, given in Definition 4, under the assumptions (H1-3). We then look at a relationship between processes with the time inversion property and the Bessel process in Section 3.4, proving that $2\rho(R.)$ is a squared Bessel process. The results of these sections culminates in a characterisation of processes with the time inversion property in terms of the infinitesimal generator in Section 3.5. Finally, Section 3.6 considers a subset of processes with the time inversion property and investigates a link between these processes and the skew product representation that will be the subject of Chapter 4.

3.2 Preliminaries and Notation

In this section, we first state some assumptions that we use throughout the chapter alongside some preliminary material. This includes a review of the Lawi semigroup density for a process enjoying the time inversion property together with the assumptions we use to make this a necessary and sufficient condition up to an h -transform. We also review the spherical coordinates notation in n -dimensions, which is more compatible with the self-similar property than Cartesian coordinates.

For the remainder of this chapter, we use the definition of the time inversion property given in Definition 4 and follow the restrictions laid down by Lawi to define the semigroup density, but extended to n dimensions. Let $R := \{(R_t : t > 0), \mathbb{P}_x\}$ be a Feller process on a state space $S = \{\mathbb{R}^n \text{ for some } n \geq 2\}$ that satisfies the assumptions (H1), (H2') and (H3) of the previous chapter extended to the state space \mathbb{R}^n . Cones which are strict subsets of \mathbb{R}^n or $\mathbb{R}^n \setminus \{0\}$ are excluded from this study.

In addition to these analogous assumptions, we also make the additional assumption on the function $\rho(x)$:

- (H4)** The function ρ in (3.1) is continuous and positive for all $x \in \mathbb{R}^n \setminus \{0\}$ vanishing only at the origin.

On account of this, we now use the notation (H1-4') to refer to assumptions (H1), (H2'), (H3) and (H4).

We also assume that this process is on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and generates the right continuous filtration $(\mathcal{F}_t^R)_{t \geq 0}$ and observe that the restrictions (H1-4') are sufficient

to invoke the work of Gallardo and Yor [2005] and Lawi [2008]. Expressly, this means that the process R given above satisfies the time inversion property of degree one if and only if its semigroup density $p_t(x, y)$ is given by

$$p_t(x, y) = \frac{1}{t^{\frac{n}{2}}} \Phi\left(\frac{x}{t^{\frac{1}{2}}}, \frac{y}{t^{\frac{1}{2}}}\right) \theta\left(\frac{y}{t^{\frac{1}{2}}}\right) \exp\left(-\rho\left(\frac{x}{t^{\frac{1}{2}}}\right) - \rho\left(\frac{y}{t^{\frac{1}{2}}}\right)\right) \quad (3.1)$$

for the same restrictions on the functions given by (2.3), (2.4) and (2.5) in the previous chapter, or it is in h -transform with a such a process. However, we also assume that $\beta > 1 - n$ in (2.4)

$$\theta(\lambda y) = \lambda^\beta \theta(y) \quad (3.2)$$

and so we omit any processes with $\beta < 1 - n$ or h -transforms of these. As these processes did not satisfy (H3) on \mathbb{R} in Chapter 2, we do not feel that this is too strong an assumption.

Initially, at least, we only burden ourselves with processes that have semigroup densities of this form since the h -transforms of Feller processes are well understood and for more detail we refer the reader to Doob [1957] and [Revuz and Yor, 2005, Chapter VIII.3].

The Spherical Coordinates Notation

The self-similar property and many of the restrictions on the semigroup density in Lawi [2008], given by (2.3), (2.4) and (2.5) are with respect to a scalar variable ($\lambda > 0$) and are therefore challenging to apply to Cartesian coordinates, which do not generally satisfy the scalar properties in higher dimensions. For this reason, in the sequel, we would like to move to spherical coordinates so we recall this change of variables here.

We state our notation here, but for more detail we refer the reader to Appendix B. For any point $y \in \mathbb{R}^n$, in spherical notation we refer to this as $y = r_y g(\underline{\phi}_y)$ where $g : \mathbb{R}^{n-1} \rightarrow S^{n-1}$ and S^{n-1} is the $(n - 1)$ -dimensional sphere on \mathbb{R}^n . The function g gives the angular part of the process. Moreover, the bijective nature of the spherical coordinates construction allows us to make an integral substitution, which we refer to as a function h satisfying $dy_1 \dots dy_n = r^{n-1} h(\underline{\phi}_y) dr d\underline{\phi}_y$. The construction of g and h are also described more fully in Appendix B.

3.3 Feller Jump Processes with the Time Inversion Property on \mathbb{R}^n

The work of Vuolle-Apiala [2012] extended a portion of the literature on the one-dimensional time inversion property to \mathbb{R}^n and provided an example class of processes enjoying the time inversion property in higher dimensions. However, these examples are restricted to processes with almost surely continuous paths; this section aims to add processes with jumps. To do this, we identify all the possible jumps of a Feller process R that enjoys the time inversion property on \mathbb{R}^n through the support of the Lévy kernel. More explicitly, we show that under the assumptions of Section 3.2, a process enjoying the time inversion property can only jump

between points that have an equal ρ -value. Here, ρ is the function given in (3.1).

We do this through a relationship between the semigroup density in Lawi [2008] and the definition of the Lévy kernel of R as the compensating measure of the point process representing the jumps of a Feller process, which we recall here. [Jacod and Shiryaev, 1987, Chapter 2 Section 1] define the Lévy kernel of a process R for any predictable f with compact support, $x \in \mathbb{R}^n$ and $t > 0$ by

$$\mathbb{E}_x \left[\sum_{s \leq t} f(R_s, R_{s-}) \mathbf{1}_{[\|R_s - R_{s-}\| > 0]} \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^n \setminus \{R_s\}} f(y, R_s) n(R_s, dy) ds \right] \quad (3.3)$$

and outside this set, they define the kernel to take the value $n(x, \{x\}) = 0$ for all $x \in \mathbb{R}^n$.

Furthermore, we also recall that the infinitesimal generator of a self-similar time-homogeneous Feller process applied to an $f \in C_0^2(\mathbb{R}^n)$ is given by (1.11) (see Section 1.7), where $l_\epsilon(z) \sim z$ in a small neighbourhood of zero and vanishes outside $B_\epsilon(0)$, the open ball of radius ϵ centred at zero. By the independence of the continuous and jump parts of the generator shown in Bass [1979], [Meyer, 1967, p. 141] has shown that for a time-homogeneous Feller process with an absolutely continuous semigroup density with respect to the Lebesgue measure

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} (f(y) - f(x) - l_\epsilon(y - x) \cdot \nabla f(x)) \frac{p_t(x, y)}{t} dy \quad (3.4)$$

$$= \int_{\mathbb{R}^n} (f(y) - f(x) - l_\epsilon(y - x) \cdot \nabla f(x)) n(x, dy). \quad (3.5)$$

The general understanding behind this is given more simply for one-dimensional Lévy processes in [Bertoin, 1998, Chapter 1]. Expressly, this implies that the quotient of the semigroup density by t integrated with respect to the Lebesgue measure converges vaguely to the Lévy kernel $n(x, \cdot)$ outside of an arbitrary neighbourhood of x .

With the Lévy kernel defined in this way, we can now determine all the jumps that are permitted by a process satisfying (H1-4') that enjoys the time inversion property almost surely. However, in order to do this, we first express the semigroup density in terms of spherical coordinates where the restrictions on the functions, given by (2.3), (2.4) and (2.5), are more easily manipulated.

Employing the techniques expressed in Section 3.2, we would like to separate the radial and unit distance parts of the integral in the way we can in two dimensions with polar coordinates and $dy_1 dy_2 = r dr d\theta$. Following the methods of Blumenson [1960] we do this by using spherical coordinates in a general n -dimensions for $n \in \mathbb{N}$, $n \geq 2$. For a particular point $y = (y_1, \dots, y_n)^T$ in \mathbb{R}^n we decompose it into its radial part, $r_y = \|y\|$, and its angular parts, $\underline{\phi}_y = (\phi_1^{(y)}, \dots, \phi_{n-1}^{(y)})^T$ as above, where, for the ease of notation, we refer to this as $y = r_y g(\underline{\phi}_y)$. For more information on this bijection, we refer the reader to Appendix B.

In order to see how this affects the semigroup density of a process with the time inversion property satisfying (H1-4'), we consider the distribution of the process. For a Borel f with

compact support and a point $x \in \mathbb{R}^n$ by (3.1)

$$\mathbb{E}_x [f(R_t)] = \int_{\mathbb{R}^n} f(y) \frac{1}{t^{\frac{n}{2}}} \Phi \left(\frac{x}{t^{\frac{1}{2}}}, \frac{y}{t^{\frac{1}{2}}} \right) \theta \left(\frac{y}{t^{\frac{1}{2}}} \right) \exp \left(-\frac{\rho(x) + \rho(y)}{t} \right) dy$$

and noting that completing the square, $\rho(x) + \rho(y) = (\sqrt{\rho(x)} - \sqrt{\rho(y)})^2 + 2\sqrt{\rho(y)\rho(x)}$, gives

$$\mathbb{E}_x [f(R_t)] = \int_{\mathbb{R}^n} f(y) \frac{1}{t^{\frac{n}{2}}} \Phi \left(\frac{x}{t^{\frac{1}{2}}}, \frac{y}{t^{\frac{1}{2}}} \right) e^{-\frac{2\sqrt{\rho(x)\rho(y)}}{t}} \theta \left(\frac{y}{t^{\frac{1}{2}}} \right) \exp \left(-\frac{(\sqrt{\rho(x)} - \sqrt{\rho(y)})^2}{t} \right) dy.$$

Thus, applying the change of coordinates and using the restrictions on the functions given by (2.3), (2.4) and (2.5)

$$\begin{aligned} \mathbb{E}_x [f(R_t)] &= \int_{\mathbb{R}^n} f(r_y g(\underline{\phi}_y)) \frac{1}{t^{\frac{n}{2}}} \Phi \left(\frac{g(\underline{\phi}_x)}{t^{\frac{1}{2}}}, \frac{r_x r_y g(\underline{\phi}_y)}{t^{\frac{1}{2}}} \right) e^{-\frac{2r_x r_y \sqrt{\rho(\underline{\phi}_x)\rho(\underline{\phi}_y)}}{t}} \theta \left(\frac{r_y g(\underline{\phi}_y)}{t^{\frac{1}{2}}} \right) \\ &\quad e^{-\frac{(r_x \sqrt{\rho(\underline{\phi}_x)} - r_y \sqrt{\rho(\underline{\phi}_y)})^2}{t}} r_y^{n-1} h(\underline{\phi}_y) dr_y d\underline{\phi}_y, \end{aligned}$$

where, in the above, $\int_{\mathbb{R}^n} = \int_0^{2\pi} (\int_0^\pi)^{n-2} \int_0^\infty$. We can make this expression slightly tidier by using a slightly different notation for Φ

$$\tilde{\Phi}_{\underline{\phi}_x, \underline{\phi}_y}(z) = \Phi \left(g(\underline{\phi}_x), z g(\underline{\phi}_y) \right) e^{-2z \sqrt{\rho(\underline{\phi}_x)\rho(\underline{\phi}_y)}},$$

which gives the distribution of the process

$$\mathbb{E}_x [f(R_t)] \tag{3.6}$$

$$= \int_{\mathbb{R}^n} f(r_y g(\underline{\phi}_y)) \tilde{\Phi}_{\underline{\phi}_x, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) e^{-\frac{(r_x \sqrt{\rho(\underline{\phi}_x)} - r_y \sqrt{\rho(\underline{\phi}_y)})^2}{t}} \frac{r_y^{n+\beta-1}}{t^{\frac{n+\beta}{2}}} h(\underline{\phi}_y) dr_y d\underline{\phi}_y. \tag{3.7}$$

Furthermore, the function θ in (3.1) can always be written $\theta(y) = \|y\|^\beta$ without changing the conditions of the semigroup density. Specifically, we can define a slightly altered $\hat{\Phi}$ that still satisfies (2.3) as

$$\hat{\Phi}(x, y) = \Phi(x, y) \theta(\hat{y})$$

where $\hat{y} = \frac{y}{\|y\|}$ is the unit part of y and therefore

$$\Phi(x, y) \theta(y) = \hat{\Phi}(x, y) \|y\|^\beta$$

where we have used (3.2).

Importantly, 3.6 reduces the dependence of the semigroup density on the radial part to $\tilde{\Phi}$, a polynomial and an exponential function; this will be invaluable in the sequel. Moreover, this permits us to determine the Lévy kernel of the process. However, before we determine the jumps precisely, we first need to show that the Lebesgue measure of the pre-image of zero under

ρ is zero.

Lemma 25. *If R is a Markov process on $S = \mathbb{R}^n$ with a semigroup density of the form (3.1) satisfying (H1-4'), then, Lebesgue-almost everywhere on S , $\rho(x) \neq 0$.*

Proof. To show that the pre-image of zero under ρ has zero measure, we consider the set of points for which $\rho(x) = 0$, which we call $A_{\rho,0} = \{y \in S : \rho(y) = 0\}$, and the semigroup density over these points. Since a probability measure always has full measure equal to one, we know that if we choose an x in the set $A_{\rho,0}$, then the semigroup density in (3.6) gives

$$\int_{\{y \in S : \rho(y)=0\}} \Phi\left(x, \frac{y}{t}\right) \frac{\|y\|^\beta}{t^{\frac{n+\beta}{2}}} dy \leq 1.$$

If we use the substitution $y = tu$, the restriction (2.5) gives the upper bound

$$t^{-\frac{n+\beta}{2}} \geq \int_{\{u \in S : \rho(u)=0\}} \Phi(x, u) \|u\|^\beta du.$$

By self-similarity and the properties of a semigroup density, $p_1(x, y) > 0$ for any x, y in the state space S and therefore, $\Phi(x, u) > 0$ for u almost everywhere in S . Since there is no dependence on t on the right hand side, we can let t tend to infinity to obtain our result.

The only problematic case is when $\beta = -n$. We consider this separately. Firstly, we show that in this case there exists an x such that $\rho(x) \neq 0$. If we assume the contrary then

$$1 = \int \Phi\left(x, \frac{y}{t}\right) \|y\|^{-n} dy = \int \Phi(x, u) \|u\|^{-n} du,$$

where we have used the same substitution. Since there is no dependence on t , this is not a Markov process and so this is a contradiction.

Considering the case of such an x , we would now like to show that the measure of the pre-image of zero is zero. We assume that it is not, and, under the same substitution, we obtain

$$\int_{\{u \in S : \rho(u)=0\}} \Phi(x, u) \|u\|^\beta e^{-\frac{\rho(x)}{t}} du \geq \epsilon$$

for some $\epsilon > 0$ since $x, y \in S$ give strictly positive values for the integrand. We can make the left hand side independent of t and letting t tend to zero this gives

$$\lim_{t \rightarrow \infty} \int_{\{u \in S : \rho(u)=0\}} \Phi(x, u) \|u\|^\beta du \geq \epsilon e^{\frac{\rho(x)}{t}} = \infty.$$

However, the measure must also always be less than or equal to one. Taking the limit as t tends to infinity

$$\lim_{t \rightarrow \infty} \int_{\{u \in S : \rho(u)=0\}} \Phi(x, u) \|u\|^\beta du \leq e^{\frac{\rho(x)}{t}} \leq 1.$$

We have arrived at a contradiction.

□

With this lemma proved, we are now in a position to comprehensively determine all the possible jumps of a Feller process with the time inversion property satisfying (H1-4'), but first we recall a few properties of Feller jump processes as applied to our process R .

Firstly, if R is a Feller process initiated at x then its semigroup density converges vaguely to the Dirac point mass function at x as t tends to zero, see [Revuz and Yor, 2005, Chapter III.2], and therefore for any open set A_x containing $\{x\}$ we have

$$\lim_{t \rightarrow 0} \int_{A_x} p_t(x, y) dy = 1 \quad (3.8)$$

by the Feller property and [Chung, 2001, p. 86 Theorem 4.3.2]. Thus, if we consider a process with a semigroup density, which in spherical coordinates has the form given in (3.6), then for any open set A containing $x = r_x g(\underline{\phi}_x)$

$$\lim_{t \rightarrow 0} \int \dots \int_{A_x} \tilde{\Phi}_{\underline{\phi}_x, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) e^{-\frac{(r_x \sqrt{\rho(\underline{\phi}_x)} - r_y \sqrt{\rho(\underline{\phi}_y)})^2}{t}} \frac{r_y^{n+\beta-1}}{t^{\frac{n+\beta}{2}}} h(\underline{\phi}_y) dr_y d\underline{\phi}_y = 1, \quad (3.9)$$

$$(3.10)$$

where we have simply taken $f = \mathbf{1}_A$ in (3.6) and simplified the expression.

Secondly, if a process is permitted to jump from a point x to a point z then (by (3.3)) every open set A_z containing that point must have positive measure assigned to it by the Lévy kernel; $n(x, A_z) > 0$. Equivalently, using the vague convergence of the quotient of the semigroup to the Lévy kernel given in (3.4) and the form of the semigroup density presented in (3.6),

$$\lim_{t \rightarrow 0} \frac{1}{t} \int \dots \int_{A_z} \tilde{\Phi}_{\underline{\phi}_x, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) e^{-\frac{(r_x \sqrt{\rho(\underline{\phi}_x)} - r_y \sqrt{\rho(\underline{\phi}_y)})^2}{t}} \frac{r_y^{n+\beta-1}}{t^{\frac{n+\beta}{2}}} h(\underline{\phi}_y) dr_y d\underline{\phi}_y > 0. \quad (3.11)$$

With these expressions regarding Feller-jump processes recalled we can now fully characterise all the possible jumps made by our process R .

Theorem 26. *Let R be a Feller process on the state space $S = \mathbb{R}^n$ satisfying assumptions (H1-4') and with a semigroup density of the form 2.2 with $\beta > 1 - n$ in (2.4). With probability one, the jumps of R from an $x \in S \setminus \{0\}$ are restricted to points $J \in S \setminus \{x, 0\}$ that satisfy $\rho(x) = \rho(J)$.*

We take this opportunity to note that Feller processes satisfying (H1-4') and enjoying the time inversion property also include the h -transforms of the set of processes we have considered above with a semigroup density of the form (3.1). However, an h -transform does not extend the support of the Lévy kernel, and therefore the possible jump destinations of the original process, provided that the original process is Feller. This can be seen by applying the h -transform to the infinitesimal generator of the process, see [Revuz and Yor, 2005, Chapter VIII.3 Proposition VIII.3.9] extended to the Feller processes in this theorem. Thus, if we prove this

theorem for processes with a semigroup density of the form (3.1), the result follows for all h -transforms of these processes. This includes all processes with the time inversion property satisfying (H1-4') by Lawi [2008].

Proof. We prove this theorem using a sandwich argument. Given a fixed arbitrary angular part and a fixed starting point, we show that the only radial part that can satisfy the jumping property (3.11) whilst still retaining the Feller property (3.9) is the point such that $\rho(J) = \rho(x)$.

A Feller process can only jump from a point x to a point J if the Lévy kernel assigns positive measure to every open neighbourhood of x . Thus, to prove the theorem we only need show that for any point J such that $\rho(J) \neq \rho(x)$ there exists some small $\delta > 0$ such that $n(x, B_\delta(J)) = 0$. This proof will rely on sandwiching the semigroup density between restrictions provided by the Feller property and restrictions provided by the jumping properties of the Lévy kernel $n(x, dy)$.

We first note that for any $y \in S$ we must have boundedness of ρ at unit distance. That is, there exists a constant $L > 0$ such that

$$\rho\left(\frac{y}{\|y\|}\right) < L. \quad (3.12)$$

This follows from property (2.5), (H4) on a closed, compact set and the requirement for a process on S : $p_t(x, y) > 0$ for some $x, y \in \mathring{S}$ for all $t > 0$.

To begin the main part of the proof, we fix an initial point x that the Lévy kernel is taken from. Intuitively, for a process with finite activity, this would be the point the process we would be jumping from. By (H3), the process is assumed to be conservative and therefore, by this and the Feller property, it cannot jump an infinite radial distance almost surely, as it does when killed, so it is only necessary for us to focus on jumps to sets less than a finite radial distance, say K , for some arbitrary $K \in (0, \infty)$ and greater than an arbitrary distance, say C , from the origin. Thus, we first check the possibilities within a set D defined as

$$D = \{y : C < r_y < K, \quad \phi_y \in [0, 2\pi) \times [0, \pi)^{n-2}\},$$

for a $C > 0$, where r_y and ϕ_y are the radial and angular parts of y respectively. Using (3.12) and (2.5), we can see that the ρ -value of any point in this set will also be bounded by $K^2 L$.

Firstly, we define the ρ -distance ball around z to be the annulus $B_{\rho, \epsilon}(z) := \{y \in D : (\sqrt{\rho(z)} - \sqrt{\rho(y)})^2 < \epsilon\}$. Since the points of equal ρ -value to z form an elliptical shape, the ball $B_{\rho, \epsilon}(z)$ can be viewed as an annulus neighbourhood of this n -dimensional ellipse. We know by (3.8) that for any arbitrary $z \in D$,

$$\lim_{u \rightarrow 0} \int \dots \int_{B_{\rho, \epsilon}(z)} \tilde{\Phi}_{\phi_z, \phi_y} \left(\frac{r_z r_y}{u} \right) \theta \left(g(\phi_y) \right) e^{-\frac{(\sqrt{\rho(z)} - \sqrt{\rho(y)})^2}{u}} \frac{r_y^{n+\beta-1}}{u^{\frac{n+\beta}{2}}} h(\phi_y) dr_y d\phi_y \leq 1,$$

where $z = r_z g(\phi_z)$. Furthermore, since we are restricted to the radial set $B_{\rho, \epsilon}(z)$ we know by the boundedness of the angular part of ρ , given in (3.12), that the part in the exponential

function must also be bounded and since $\beta > 1 - n$

$$\lim_{u \rightarrow 0} u^{-\frac{n+\beta}{2}} e^{-\frac{\epsilon}{u}} \min_{r \in B_{\rho, \epsilon}(z)} \{r^{n+\beta-1}\} \int \dots \int_{B_{\rho, \epsilon}(z)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{u} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y \leq 1.$$

Thus, for any choice of $\epsilon > 0$ and any $z \in D$,

$$\lim_{u \rightarrow 0} e^{-\frac{\epsilon}{u}} \int \dots \int_{B_{\rho, \epsilon}(z)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_z r_y}{u} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y = 0,$$

where we have used the strict inequality in the definition of $B_{\rho, \epsilon}(z)$. Moreover, we can now choose any $r_x \in (C, K)$ and by making the substitution $u = \frac{tr_z}{r_x}$

$$\lim_{t \rightarrow 0} e^{-\frac{\epsilon}{t} \frac{r_x}{r_z}} \int \dots \int_{B_{\rho, \epsilon}(z)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y = 0.$$

Equivalently, since $r_z > C > 0$, then for any finite $r_x \in (C, K]$ we can choose a $\delta = \epsilon \frac{r_x}{r_z} > 0$ such that

$$\lim_{t \rightarrow 0} e^{-\frac{\delta}{t}} \int \dots \int_{B_{\rho, \delta \frac{r_x}{r_z}}(z)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y = 0. \quad (3.13)$$

and since we could choose $\epsilon > 0$ arbitrarily this holds for any $\delta > 0$ and $r_x, r_z \in (C, K)$ and this corresponds to the radial parts of points $x, z \in D$ that have the same angular component.

However, we would in fact like to prove the stronger claim on the integral over the entirety of D : for all $\delta > 0$ and $r_x \in (C, K)$

$$\lim_{t \rightarrow 0} e^{-\frac{\delta}{t}} \int \dots \int_D \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y = 0. \quad (3.14)$$

If we fix $\delta > 0$, then by the compactness of the closure of D we can write this set as a finite union of balls, or open rings, with ρ -size $\delta \frac{C}{K} > 0$ that cover the set. Namely,

$$D \subset \bigcup_{i=1}^N B_{\rho, \delta \frac{C}{K}}(z_i) \subset \bigcup_{i=1}^N B_{\rho, \delta \frac{r_{z_i}}{r_x}}(z_i)$$

for a finite number of points $z_i \in D$ that all have angular part $\underline{\phi}_{z_i} = \underline{\phi}_x$. The second subset follows because $\delta \frac{r_{z_i}}{r_x} > \delta \frac{C}{K}$ for all $z_i \in D$. Therefore, our integral is bounded above by the sum of the integral over each of these finite balls

$$\begin{aligned} & e^{-\frac{\delta}{t}} \int \dots \int_D \tilde{\Phi}_{\underline{\phi}_x, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y \\ & \leq \sum_{i=1}^N e^{-\frac{\delta}{t}} \int \dots \int \int_{B_{\rho, \delta \frac{r_{z_i}}{r_x}}(z_i)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y \end{aligned}$$

where the less than or equal to sign is guaranteed by the positivity of the integrand. Moreover,

since $z_i, x \in D$, each of these balls satisfy (3.13). Since our sum is finite we have also expanded this out to see that we have a sum of finitely many integrals whose limit is zero. Thus, by (3.13) and the summation of limits the overall limit is zero as claimed in (3.14).

With claim (3.14) confirmed, we now utilise this to show that the Lévy kernel $n(x, \cdot)$ cannot possibly assign measure to any set in $B_{\rho, M}(r_x)$ whose ρ -distance is greater than some $\epsilon > 0$ from x . Without loss of generality, we assume that the angular part of x is $\underline{\phi}_z$ as used above.

Employing our expression for the Lévy kernel given in (3.11) for the set $D \setminus B_{\rho, \epsilon}(x)$ for a fixed $\epsilon > 0$ and a $x \in D$ with the same angular part as z

$$\begin{aligned} n(x, D \setminus B_{\rho, \epsilon}(x)) \\ = \lim_{t \rightarrow 0} \frac{1}{t} \int \dots \int_{D \setminus B_{\rho, \epsilon}(x)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) e^{-\frac{(r_x \sqrt{\rho(\underline{\phi}_x)} - r_y \sqrt{\rho(\underline{\phi}_y)})^2}{t}} r_y^{\frac{n+\beta-1}{2}} h(\underline{\phi}_y) dr_y d\underline{\phi}_y, \end{aligned}$$

and using the restriction on $B_{\rho, \epsilon}(x)$ to provide an upper bound on the measure

$$\begin{aligned} n(x, D \setminus B_{\rho, \epsilon}(x)) \\ < \lim_{t \rightarrow 0} e^{-\frac{\epsilon}{t}} \max_{r \in D \setminus B_{\rho, \epsilon}(x)} \{r^{n+\beta-1}\} t^{-1-\frac{n+\beta}{2}} \int \dots \int_{D \setminus B_{\rho, \epsilon}(x)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y. \end{aligned}$$

Moreover, by the strictness of the inequality in $B_{\rho, \epsilon}(r_x)$ the exponential term dominates leaving

$$\begin{aligned} n(x, D \setminus B_{\rho, \epsilon}(r_x)) \\ < \lim_{t \rightarrow 0} e^{-\frac{\epsilon}{t}} \int \dots \int_{D \setminus B_{\rho, \epsilon}(x)} \tilde{\Phi}_{\underline{\phi}_z, \underline{\phi}_y} \left(\frac{r_x r_y}{t} \right) \theta \left(g(\underline{\phi}_y) \right) h(\underline{\phi}_y) dr_y d\underline{\phi}_y \end{aligned}$$

and this vanishes by (3.14). Thus, the Lévy kernel can only assign measure to points in $B_{\rho, \epsilon}(x)$ for an arbitrary $\epsilon > 0$.

Applying this same argument to every angular part, one can see that the Lévy kernel can only possibly assign measure to points y such that $\rho(x) = \rho(y)$ whilst still retaining the Feller properties. □

3.4 Processes with the Time Inversion Property and the Bessel Process

In Chapter 2, we showed that for any process R with a semigroup density of the form (2.2) on \mathbb{R} , $2\rho(R_\cdot)$ was a squared Bessel process. This section extends this result to \mathbb{R}^n .

Remark 27. *We note that by Lemma 25, the positivity restriction holds for x almost everywhere on the state space and therefore the positivity assumption is only the limited assumption extending the support from almost everywhere to the entire space minus the origin. The assumption is limited further since we are considering a semigroup density with respect to the Lebesgue*

measure. The continuity restriction is also a limited one given that we have already made the assumption that $p_t(x, y) \in C^2$.

To show the main result of this section, we first show that the process $(\rho(R.)_t)_{t \geq 0}$ does in fact retain the Markov property from R (Lemma 28) and, once we have this result, we show that the process satisfies both self-similarity and positivity. As squared Bessel processes with time-scaling are the only positive 1-self-similar Markov processes with continuous paths, the result follows from this almost immediately.

To begin with, however, we show the Markov property of the process $(\rho(R.)_t)_{t \geq 0}$ by using the expression of the semigroup density and the restrictions (H1-4').

Lemma 28. *If R is a Feller process on \mathbb{R}^n with a semigroup density of the form (3.1) satisfying the assumptions (H1-4') and ρ is the exponential function in the form of this semigroup then the process given by $\rho(R_t)$ is Markov.*

If ρ was injective, then we would have our result from the Markov property of R . Thus, this lemma reduces to showing that any two points z and w that are mapped by ρ to the same point give the same distribution of $\rho(R.)$ regardless of whether R is initiated at z or w . We prove this by using (H3) and the form of the semigroup density to show that the Laplace transform of the distribution of $\rho(R.)$ is independent of whether R is started at z or w .

Proof. In the following we use: \mathbb{P}_x to denote the measure of R initiated at x ; $\tilde{\mathbb{P}}_{\rho(x)}$ to denote the measure of $\rho(R.)$ initiated at $\rho(x)$ and $\hat{\mathbb{P}}_{\sqrt{\rho(x)}}$ to denote the measure of $\sqrt{\rho(R.)}$ initiated at $\sqrt{\rho(x)}$. We also note the equivalence

$$\hat{\mathbb{E}}_{\sqrt{\rho(x)}} \left[f \left(\left(\sqrt{\rho(R_t)} \right)^2 \right) \right] = \tilde{\mathbb{E}}_{\rho(x)} [f(\rho(R_t))] = \mathbb{E}_x [f(\rho(R_t))].$$

In order to show that $\rho(R_t)$ is Markov, we would like to show that for every Borel function f ,

$$\tilde{\mathbb{E}}_x \left[f(\rho(R_t)) | \mathcal{F}_s^{\rho(R.)} \right] = \tilde{\mathbb{E}}_x [f(\rho(R_t)) | \sigma(\rho(R_s))]$$

or equivalently, by time homogeneity, that

$$\tilde{\mathbb{E}}_x \left[f(\rho(R_t)) | \mathcal{F}_s^{\rho(R.)} \right] = \tilde{\mathbb{E}}_{\rho(R_s)} [f(\rho(R_{t-s}))], \quad (3.15)$$

where $\mathcal{F}_s^{\rho(R.)} = \sigma(\rho(R_u) : u \leq s)$ is the σ -algebra generated by $\rho(R_u)$ for all $u \leq s$. However, by the Markov and time homogeneous properties of R together with the bijective property of the square root function, we know that

$$\hat{\mathbb{E}}_x \left[f(\sqrt{\rho(R_{t-s}))} \circ \theta_s | \mathcal{F}_s^{\sqrt{\rho(R.)}, \frac{R_s}{\sqrt{\rho(R_s)}}} \right] = \mathbb{E}_{R_s} [f(\sqrt{\rho(R_{t-s}))}] = \hat{\mathbb{E}}_{\sqrt{\rho(R_s)}, \frac{R_s}{\sqrt{\rho(R_s)}}} [f(\sqrt{\rho(R_{t-s}))}],$$

where we have used the Borel property of $f \circ \rho$, which is bestowed by the Borel property of f and the continuity of ρ (H4) and the slight abuse of notation in the subscript of the final expectation to refer to the starting points of the radial and angular parts. The second equality is true because all the information about R is exactly equivalent to the information given by the ρ -radial $\sqrt{\rho(R_s)}$ and ρ -angular $\frac{R_s}{\sqrt{\rho(R_s)}}$ parts, since the transform $R_s \rightarrow \left(\frac{R_s}{\sqrt{\rho(R_s)}}, \sqrt{\rho(R_s)} \right)$ is bijective.

Thus, if we can show that this is independent of the angular part we have

$$\hat{\mathbb{E}}_x \left[f(\sqrt{\rho(R_t)}) | \mathcal{F}_s^{\sqrt{\rho(R_\cdot)}} \right] = \hat{\mathbb{E}}_{\sqrt{\rho(R_s)}} \left[f(\sqrt{\rho(R_{t-s})}) \right],$$

which by the bijective nature of square root on \mathbb{R}^+ gives the Markov property (3.15). We show the independence of the radial part by showing that

$$\mathbb{E}_z [f(\rho(R_{t-s}))] = \mathbb{E}_w [f(\rho(R_{t-s}))]$$

for all z, w such that $\rho(z) = \rho(w)$, or equivalently $\sqrt{\rho(z)} = \sqrt{\rho(w)}$, i.e. that the distribution of the process only depends on the ρ -value of the current position and since the square root function is bijective this is sufficient. We show this in several steps.

First, we can express the semigroup density of R as (3.1) for all the usual conditions on the component functions (2.3), (2.4) and (2.5). Using this, we would like to show that the following integrals are equal for all z, w such that $\rho(z) = \rho(w)$

$$\int_0^\pi \dots \int_0^{2\pi} \Phi \left(z, \frac{\sqrt{u}g(\phi)}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{u}g(\phi))}{(\rho(g(\phi)))^{\frac{n+\beta}{2}}} h(\phi) d\phi \quad (3.16)$$

$$= \int_0^\pi \dots \int_0^{2\pi} \Phi \left(w, \frac{\sqrt{u}g(\phi)}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{u}g(\phi))}{(\rho(g(\phi)))^{\frac{n+\beta}{2}}} h(\phi) d\phi \quad (3.17)$$

almost everywhere with respect to u .

We can do this using the condition (H3), R is conservative, and the properties of the semigroup density. By (H3); our assumption $\rho(z) = \rho(w)$ and the condition on ρ , given in (2.5), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi \left(z, \frac{y}{t} \right) \frac{\theta(y)}{t^{\frac{n+\beta}{2}}} \exp \left(-\frac{\rho(z) + \rho(y)}{t} \right) dy &= 1 = \int_{\mathbb{R}^n} \Phi \left(w, \frac{y}{t} \right) \frac{\theta(y)}{t^{\frac{n+\beta}{2}}} \exp \left(-\frac{\rho(w) + \rho(y)}{t} \right) dy, \\ \int_{\mathbb{R}^n} \Phi \left(z, \frac{y}{t} \right) \theta(y) \exp \left(-\frac{\|y\|^2 \rho(\hat{y})}{t} \right) dy &= \int_{\mathbb{R}^n} \Phi \left(w, \frac{y}{t} \right) \theta(y) \exp \left(-\frac{\|y\|^2 \rho(\hat{y})}{t} \right) dy, \end{aligned}$$

where we have simplified each side. Many of the restrictions on the semigroup density, and indeed self-similarity, are scalar in nature so we would like to get the integral in a form where we can exploit this. Taking spherical coordinates as explained in Section 3.2 using

$dy = r^{n-1}drh(\phi)d\phi$; $y = rg(\phi)$ and ϕ as notation for the $n - 1$ angular parts of y we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \Phi \left(z, \frac{rg(\phi)}{t} \right) \theta(rg(\phi)) \exp \left(-\frac{r^2 \rho(g(\phi))}{t} \right) r^{n-1} dr h(\phi) d\phi \\ &= \int_{\mathbb{R}^n} \Phi \left(w, \frac{rg(\phi)}{t} \right) \theta(rg(\phi)) \exp \left(-\frac{r^2 \rho(g(\phi))}{t} \right) r^{n-1} dr h(\phi) d\phi. \end{aligned}$$

By (H4), $\rho > 0$ outside the origin and therefore the function is positive at any point on the ellipse that is ρ -distance one from the origin. This implies that $\rho(g(\phi)) > 0$ for any $\phi \in S^{n-1}$ so inside the radial part we can now make the substitution $r\sqrt{\rho(g(\phi))} = \sqrt{ut}$

$$\begin{aligned} & \int_0^{2\pi} \dots \int_0^\infty \Phi \left(z, \frac{\sqrt{ug(\phi)}}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{ut}g(\phi))}{(\rho(g(\phi)))^{\frac{n+\beta}{2}}} \exp(-ut) \frac{u^{\frac{n-2}{2}}}{2} t^n du h(\phi) d\phi \\ &= \int_0^{2\pi} \dots \int_0^\infty \Phi \left(w, \frac{\sqrt{ug(\phi)}}{\sqrt{\rho(g(\phi))}t} \right) \frac{\theta(\sqrt{ug(\phi)})}{(\rho(g(\phi)))^{\frac{n+\beta}{2}}} \exp(-ut) \frac{u^{\frac{n-2}{2}}}{2} t^n du h(\phi) d\phi. \end{aligned}$$

Since the integrand is non-negative for all points in the state space we can use the Fubini-Tonelli Theorem to exchange the integrals

$$\begin{aligned} & \int_0^\infty \left(\int_0^\pi \dots \int_0^{2\pi} \Phi \left(z, \frac{\sqrt{ug(\phi)}}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{ug(\phi)})}{(\rho(g(\phi)))^{\frac{n+\beta}{2}}} h(\phi) d\phi \right) \frac{u^{\frac{n-2}{2}}}{2} \exp(-ut) du \\ &= \int_0^\infty \left(\int_0^\pi \dots \int_0^{2\pi} \Phi \left(w, \frac{\sqrt{ug(\phi)}}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{ug(\phi)})}{(\rho(g(\phi)))^{\frac{n+\beta}{2}}} h(\phi) d\phi \right) \frac{u^{\frac{n-2}{2}}}{2} \exp(-ut) du. \end{aligned}$$

Since this holds for all $t \in (0, \infty)$, then by the uniqueness of Laplace transforms on $[0, \infty)$ we know that the angular parts inside the brackets are equal almost everywhere by Lerch's Theorem. Expressly, we have validated the claim made in (3.16).

In order to show the Markov property for the process $\rho(R_t)$, as previously mentioned, we would like to show that for any Borel compactly supported f and any w such that $\rho(w) = \rho(z)$ we have that

$$\mathbb{E}_z [f(\rho(R_t))] = \mathbb{E}_w [f(\rho(R_t))]$$

that is, the distribution only depends on the ρ -value of the starting point. This expectation is finite for all $t < \infty$ by the compact support of f . This equality is what we show now.

Since R satisfies (H1-3') and has the time inversion property, the equation can be written with the semigroup density of the process

$$\begin{aligned} \mathbb{E}_z [f(\rho(R_t))] &= \int_{\mathbb{R}^n} f(\rho(y)) \Phi \left(z, \frac{y}{t} \right) \frac{\theta(y)}{t^{\frac{n+\beta}{2}}} \exp \left(-\frac{\rho(z) + \rho(y)}{t} \right) dy \\ &= \int_{\mathbb{R}^n} f(\|y\|^2 \rho(\hat{y})) \Phi \left(z, \frac{\|y\|\hat{y}}{t} \right) \frac{\theta(\|y\|\hat{y})}{t^{\frac{n+\beta}{2}}} \exp \left(-\frac{\rho(z) + \|y\|^2 \rho(\hat{y})}{t} \right) dy \end{aligned}$$

where we have used the property of ρ given by (2.5) and \hat{y} is used as notation for the normalised

part of y this time given by $\frac{y}{\|y\|}$.

We now express this integral in the spherical coordinates,

$$\begin{aligned} \mathbb{E}_z [f(\rho(R_t))] \\ = \int_0^{2\pi} \dots \int_0^\infty f(r^2 \rho(g(\phi))) \Phi \left(z, \frac{rg(\phi)}{t} \right) \frac{\theta(rg(\phi))}{t^{\frac{n+\beta}{2}}} \exp \left(-\frac{\rho(z) + r^2 \rho(g(\phi))}{t} \right) r^{n-1} dr h(\phi) d\phi. \end{aligned}$$

Inside the radial part we now make the same substitution as previously $r\sqrt{\rho(g(\phi))} = \sqrt{ut}$ using the fact that $\rho(g(\phi)) > 0$ for all $\phi \in S^{n-1}$

$$\begin{aligned} \mathbb{E}_z [f(\rho(R_t))] \\ = \int_0^{2\pi} \dots \int_0^\infty f(u) \Phi \left(z, \frac{\sqrt{u}g(\phi)}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{u}tg(\phi))}{(\rho(g(\phi))t)^{\frac{n+\beta}{2}}} \exp \left(-\frac{\rho(z) + ut^2}{t} \right) \frac{u^{\frac{n-2}{2}}}{2} t^n du h(\phi) d\phi. \end{aligned}$$

Employing the Fubini-Tonelli Theorem once again, which we can do because we have non-negative terms in the integrand, and rearranging yields

$$\begin{aligned} \mathbb{E}_z [f(\rho(R_t))] \\ = \int_0^\infty \dots \int_0^{2\pi} f(u) \Phi \left(z, \frac{\sqrt{u}g(\phi)}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{u}tg(\phi))}{(\rho(g(\phi))t)^{\frac{n+\beta}{2}}} h(\phi) d\phi \frac{u^{\frac{n-2}{2}}}{2} e^{-\frac{\rho(z)+ut^2}{t}} t^n du \\ = \int_0^\infty f(u) \left(\int_0^\pi \dots \int_0^{2\pi} \Phi \left(z, \frac{\sqrt{u}g(\phi)}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{u}tg(\phi))}{(\rho(g(\phi))t)^{\frac{n+\beta}{2}}} h(\phi) d\phi \right) \frac{u^{\frac{n-2}{2}}}{2} e^{-\frac{\rho(z)+ut^2}{t}} \frac{t^{n+\beta}}{t^{\frac{n+\beta}{2}}} du. \end{aligned}$$

Then by our previous claim (3.16) for the part inside the brackets and using the assumption $\rho(z) = \rho(w)$

$$\begin{aligned} \mathbb{E}_z [f(\rho(R_t))] \\ = \int_0^\infty \dots \int_0^{2\pi} f(u) \Phi \left(w, \frac{\sqrt{u}g(\phi)}{\sqrt{\rho(g(\phi))}} \right) \frac{\theta(\sqrt{u}tg(\phi))}{(\rho(g(\phi))t)^{\frac{n+\beta}{2}}} h(\phi) d\phi \frac{u^{\frac{n-2}{2}}}{2} \exp \left(-\frac{\rho(w) + ut^2}{t} \right) t^n du. \end{aligned}$$

Finally, if we reverse the substitution and go back to Cartesian coordinates

$$\begin{aligned} \mathbb{E}_z [f(\rho(R_t))] &= \int_{\mathbb{R}^n} f(r^2 \rho(g(\phi))) \Phi \left(w, \frac{rg(\phi)}{t} \right) \frac{\theta(rg(\phi))}{t^{\frac{n+\beta}{2}}} e^{-\frac{\rho(w)+r^2\rho(g(\phi))}{t}} r^{n-1} dr h(\phi) d\phi \\ &= \int_{\mathbb{R}^n} f(\rho(y)) \Phi \left(w, \frac{y}{t} \right) \frac{\theta(y)}{t^{\frac{n+\beta}{2}}} \exp \left(-\frac{\rho(w) + \rho(y)}{t} \right) dy \\ &= \mathbb{E}_w [f(\rho(R_t))] \end{aligned}$$

and we are done. □

Since multiplication by a constant is injective, it is clear that the Markov property for $K\rho(R_t)$ for any $K > 0$ follows from this result. Thus, we are now in a position to show that $2\rho(R_t)$ is equivalent in distribution to a squared Bessel process.

Theorem 29. *Let R be a self-similar Feller process with a semigroup density of the form (3.1), with $\beta > 1 - n$ in (2.4), satisfying assumptions (H1-4'); initiated at $r \in \mathbb{R}^n$ and killed at the origin. $2\rho(R_\cdot)$ is equivalent in law to $BESQ^\delta(2\rho(r))$ up to a possible time-scaling.*

Proof. In order to prove the theorem, we prove that $\rho(R_\cdot)$ is a positive 1-self-similar Markov process with continuous paths and therefore it is a Bessel squared process that is possibly time-scaled. This follows because it has been shown in Lamperti [1972] that the only 1-pssMps with continuous paths are squared Bessel processes.

From Lemma 28 above, we know that $\rho(R_t)$ is Markov and therefore we are only required to show that it is 1-self-similar, Feller, positive and has continuous paths. The positive part comes from the positive restriction on the function $\rho(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ (H4) and we know that $\rho(0) = 0$ and this is the only point where it vanishes on the state space. The Feller property follows from the continuity of $\rho(\cdot)$ (H4) and the fact that $f \circ \rho(\cdot) \in C_0$ for any $f \in C_0$ as a result. We now try to prove the self-similar property for this process. If we assume that $V_t = (\rho(R_\cdot)_{t \geq 0})$ is initiated at a point $x = \rho(r)$ then for any Borel f and $c > 0$ by using the self-similarity of R together with (2.5),

$$\mathbb{E}_x [f(V_t)] = \mathbb{E}_{cr} \left[f \left(\frac{1}{c^2} \rho(R_{tc^2}) \right) \right] = \mathbb{E}_{c^2 z} \left[f \left(\frac{1}{c^2} V_{tc^2} \right) \right].$$

Letting $c_1 = c^2 > 0$, this shows 1-self-similarity.

Furthermore, on account of the fact that, other than killing, a process can only jump to and from points that have equal ρ -values by Theorem 26 we know that V has continuous paths almost surely.

Since $\rho(R_\cdot)$ is a 1-self-similar Feller process with continuous paths on \mathbb{R}^+ we know that it is a constant multiplied by a Bessel squared process that is possibly time changed.

□

Remark 30. *Since we have the Markov property and the dependence of the semigroup density of $\rho(R_\cdot)$ on only the ρ -value of the process, the result in Theorem 29 follows from the result in the previous chapter. In order to see this, it is only necessary to show that $\sqrt{\rho(R_\cdot)}$ has the time inversion property whenever R has the time inversion property. The result then follows from our result on processes with the time inversion property on \mathbb{R}^+ .*

To demonstrate the time inversion property for $\sqrt{\rho(R_\cdot)}$, we consider the distribution of the process $\sqrt{\rho(Y^{(x)}_\cdot)}$ and since ρ is continuous (H4), and therefore $f \circ \rho$ is Borel for any Borel f , by the time inversion property of R we have

$$\mathbb{E}_a \left[f \left(\sqrt{\rho(Y_h^{(x)})} \right) \right] = \mathbb{E}_x \left[f \left(\sqrt{\rho((t+h)R_{\frac{1}{t+h}})} \right) | tR_{\frac{1}{t}} = a \right],$$

where we know that $Y^{(x)}$ is a time homogeneous Markov process. Then, using the restriction

on ρ given by (2.5) and the decomposition of R into radial and angular parts

$$\mathbb{E}_a \left[f \left(\sqrt{\rho(Y_h^{(x)})} \right) \right] = \mathbb{E}_x \left[f((t+h)\sqrt{\rho(R_{\frac{1}{t+h}})}) \middle| t\sqrt{\rho(R_{\frac{1}{t}})} = \sqrt{\rho(a)}; \frac{R_t}{\sqrt{\rho(R_{\frac{1}{t}})}} = \frac{a}{\sqrt{\rho(a)}} \right].$$

However, by Lemma 28, $\rho(R_\cdot)$ has a semigroup density that is independent of the angular part and therefore the radial part of the process is independent of the angular part. Thus,

$$\mathbb{E}_a \left[f \left(\sqrt{\rho(Y_h^{(x)})} \right) \right] = \mathbb{E}_{\rho(x)} \left[f((t+h)\sqrt{\rho(R_{\frac{1}{t+h}})}) \middle| t\sqrt{\rho(R_{\frac{1}{t}})} = \sqrt{\rho(a)} \right],$$

and so, since the left hand side determines a time homogeneous Feller process, $\sqrt{\rho(R_t)}$ has the time inversion property and its inverted process is given by $\sqrt{\rho(Y_t^{(x)})}$. Furthermore, since the semigroup density of R is absolutely continuous with respect to the Lebesgue measure so is the semigroup density of $\sqrt{\rho(R_\cdot)}$ and the conservative property on the state space also follows. As $\sqrt{\rho(R_\cdot)}$ is a Feller process on \mathbb{R}^+ satisfying (H1-3') and enjoying the time inversion property we know from Chapter 2 that $\sqrt{\rho(R_\cdot)}$ is a, possibly time-scaled, Bessel process. Thus, we have the result of Theorem 29 up to time-scaling.

Moreover, since $\sqrt{\rho(Y_t^{(x)})}$ in this case is also a process with the time inversion property on \mathbb{R}^+ , the results of Chapter 2 imply that $\sqrt{\rho(Y_t^{(x)})}$ is a, possibly time-scaled, Bessel process in the wide sense.

3.5 Characterising a Process Enjoying the Time Inversion Property in n -Dimensions

Sections 3.3 and 3.4 have provided two significant restrictions that the process R must satisfy in order to enjoy the time inversion property on \mathbb{R}^n . In this section we would like to utilise these restrictions to characterise the process in terms of its infinitesimal generator.

For the moment, we focus on the processes satisfying (H1-4') that have the time inversion property and a semigroup density of the form 2.2 as the other processes enjoying the time inversion property result from h -transforms of these. We also denote the function $\hat{\rho}(x) = \sqrt{2\rho(x)}$ to make the notation slightly simpler.

To obtain a characterisation of a process with a semigroup density of the form (3.1) through its infinitesimal generator we focus upon three major restrictions.

1. The 2-self-similarity of the process. This is provided by the representation of the semigroup density.
2. The restriction on the possible Lévy kernels of the process given by Theorem 26.
3. $\hat{\rho}(R_t)$ is a Bessel process on \mathbb{R}^+ initiated at $\hat{\rho}(x)$.

We note that here we only identify the generator on $\mathbb{R}^n \setminus \{0\}$ without discussing the domain of the process. Thus, we are only concerned with the behaviour of the process away from zero.

Theorem 31. *Let R be a Feller process on \mathbb{R}^n with a semigroup density of the form (3.1) satisfying (H1-4'). The infinitesimal generator of R applied to a function $f \in C_0^2(\mathbb{R}^n \setminus \{0\})$ is given by*

$$\mathcal{A}_R f(x) = \sum_{i=1}^n \mu_i(x) \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^n \frac{\sigma_{ij}}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} + \int_{A_{\rho,x}} (f(y) - f(x) - l_\epsilon(y-x) \cdot \nabla f(x)) n(x, dy), \quad (3.18)$$

where the support on $n(x, \cdot)$ is confined to a subset of the set $A_{\rho,x} := \{y \in \mathbb{R}^n : \rho(x) = \rho(y)\}$ and here $\Sigma := \{\sigma_{ij}\}_{i,j=1}^n$ for a positive definite matrix Σ that also satisfies

$$\sum_{i,j=1}^n \frac{\sigma_{ij}}{2} \frac{\partial(2\rho(x))}{\partial x_i} \frac{\partial(2\rho(x))}{\partial x_j} = 4\rho(x) \quad (3.19)$$

and a μ that satisfies

$$\sum_{i=1}^n \mu_i(x) \frac{\partial(2\rho(x))}{\partial x_i} + \sum_{i,j=1}^n \frac{\sigma_{ij}}{2} \frac{\partial^2(2\rho(x))}{\partial x_i \partial x_j} = 2\nu + 2. \quad (3.20)$$

Here ν is the index of the Bessel process whose distribution is given by $\hat{\rho}(R_t)$.

Proof. It was discussed in Section 1.7 that the self-similar restriction means that the infinitesimal generator of R is of the form (1.11), where l_ϵ vanishes outside of a neighbourhood of zero; $\mu_i(\frac{x}{c}) = c\mu_i(x)$ and $c^2 n(cx, dy) = n(x, dy)$ for all $1 \leq i \leq n$ and $c > 0$ by the condition $c^{-2} \mathcal{A}_R f(c \cdot)(\frac{x}{c}) = \mathcal{A}_R f(x)$. However, the restriction on the jumps given by Theorem 26 reduces this further to the expression in the theorem given by (3.18).

Additionally, we know that $\hat{\rho}(R_t)$ is a Bessel process and so the generator must reflect this. This implies that the generator of the process $\hat{\rho}(R_t)$ initiated at $x \in \mathbb{R}^n$ must be equivalent to the generator of a Bessel process initiated at $\hat{\rho}(x)$ or, under application to a $g \in C^2(\mathbb{R}^+)$,

$$\mathcal{A}_r g(\hat{\rho}(x)) = \mathcal{A}_R(g \circ \hat{\rho})(x).$$

where \mathcal{A}_r is the infinitesimal generator of a Bessel process. Utilising the chain rule we can see that

$$\frac{\partial g(\hat{\rho}(x))}{\partial x_i} = g'(\hat{\rho}(x)) \frac{\partial(\hat{\rho}(x))}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 g(\hat{\rho}(x))}{\partial x_i \partial x_j} = g''(\hat{\rho}(x)) \frac{\partial(\hat{\rho}(x))}{\partial x_i} \frac{\partial(\hat{\rho}(x))}{\partial x_j} + g'(\hat{\rho}(x)) \frac{\partial^2(\hat{\rho}(x))}{\partial x_i \partial x_j}$$

and therefore by the support of the jumping measure given in Theorem 26

$$\begin{aligned} \mathcal{A}_R(g \circ \hat{\rho})(x) &= g'(\hat{\rho}(x)) \left[\sum_{i=1}^n \mu_i(x) \frac{\partial(\hat{\rho}(x))}{\partial x_i} + \sum_{i,j=1}^n \frac{\sigma_{ij}}{2} \frac{\partial^2(\hat{\rho}(x))}{\partial x_i \partial x_j} + \int_{\{y: \rho(x)=\rho(y)\}} l_\epsilon(y) \rho'(x) n(x, dy) \right] \\ &\quad + g''(\hat{\rho}(x)) \left[\sum_{i,j=1}^n \frac{\sigma_{ij}}{2} \frac{\partial(\hat{\rho}(x))}{\partial x_i} \frac{\partial(\hat{\rho}(x))}{\partial x_j} \right]. \end{aligned}$$

We know that this is equivalent to the infinitesimal generator of a Bessel process given by

$$\mathcal{A}_r g(\hat{\rho}(x)) = \frac{2\nu + 1}{2\hat{\rho}(x)} g'(\hat{\rho}(x)) + \frac{1}{2} g''(2\rho(x)).$$

Thus, by equating the left hand side and right hand side of the equation with respect to the coefficients of the derivatives of g

$$\int_{\{y: \rho(x)=\rho(y)\}} l_\epsilon(y) \rho'(x) n(x, dy) + \sum_{i=1}^n \mu_i(x) \frac{\partial(\hat{\rho}(x))}{\partial x_i} + \sum_{i,j=1}^n \frac{\sigma_{ij}}{2} \frac{\partial^2(\hat{\rho}(x))}{\partial x_i \partial x_j} = \frac{2\nu + 1}{2\hat{\rho}(x)}$$

and

$$\sum_{i,j=1}^n \sigma_{ij} \frac{\partial(\hat{\rho}(x))}{\partial x_i} \frac{\partial(\hat{\rho}(x))}{\partial x_j} = 1.$$

□

3.6 The Class \mathcal{U} and the Skew Product Representation

In this section, we restrict our horizon to a subset of Feller processes with the time inversion property that avoid zero almost surely. This leads on to a strong link with the skew product representation, which will be the subject of Chapter 4.

Subsection 3.6.1 considers a subset \mathcal{V} of processes with a semigroup density of the form (3.1) and shows that this subset is simply a linear mapping of an even smaller set of processes \mathcal{U} whose properties can be more easily deduced. We also show that any linear maps of this form preserve the time inversion property. In Subsection 3.6.2, we consider examples of this restricted set of processes and show that, to our knowledge, all known examples of processes with the time inversion property fall into this class. Finally, Subsection 3.6.3 characterises the set \mathcal{U} through the infinitesimal generator, therefore characterising the set \mathcal{V} and shows that, under a further restriction, this set of processes satisfies the skew product representation.

We first consider a subset of processes with a semigroup density of the form (3.1), where,

in addition to (H4), ρ satisfies

$$\rho^Z(x) = \sum_{i,j=1}^n \frac{\alpha_{ij}}{2} x_i x_j. \quad (3.21)$$

for $\alpha_{ij} \in \mathbb{R}$ and $x^T = (x_1, \dots, x_n) \in \mathbb{R}^n$, where $A := \{\alpha_{ij}\}_{i,j=1}^n$ is positive definite by (H4).

This is the most general representation of ρ we consider in this section and we denote Z to be a Feller process on \mathbb{R}^n , satisfying (H1-4'), which has a semigroup density of the form (3.1), with a function ρ as above. To make the notation simpler, we also refer to this general form of the function ρ as ρ^Z .

We also introduce a subset of this class of processes such that ρ satisfies

$$\rho^X(x) = \frac{\|x\|^2}{2}. \quad (3.22)$$

Again, to make the notation simpler we also refer to this form of the function ρ as ρ^X and we denote X to be a Feller process on \mathbb{R}^n belonging to this class. From now on, we refer to these classes as \mathcal{V} and \mathcal{U} as follows

$$\begin{aligned} \mathcal{V} &:= \{Z_t \in \mathbb{R}^n : \text{semigroup form (3.1) satisfying (H1-4') and } \rho \text{ is (3.21)}\}, \\ \mathcal{U} &:= \{X_t \in \mathbb{R}^n : \text{semigroup form (3.1) satisfying (H1-4') and } \rho \text{ is (3.22)}\}. \end{aligned}$$

3.6.1 A Subset of Processes Enjoying the Time Inversion Property

We first explore a surjective mapping that takes processes in \mathcal{U} to processes in \mathcal{V} .

Lemma 32. *Let $Z := (Z_t)_{t \geq 0}$ be a Feller process, which is a member of \mathcal{V} and avoids zero almost surely. There exists an invertible matrix M such that $Z_t = M \cdot X_t$ and X is also a Feller process, which avoids zero almost surely, that is a member of \mathcal{U} .*

Furthermore, for Z as above, the process $2\rho^Z(Z_t) = \|X_t\|^2 = 2\rho^X(X_t) \stackrel{(d)}{=} Q_t$, where Q is a squared Bessel process.

Proof. We prove this by utilising the fact that the Markov property is preserved by an invertible matrix and therefore, the process $M \cdot X$ is also a Markov process for a Markov process X . We can then also use the effect of the matrix on the semigroup density to prove that X is also a process with a semigroup density of the form (3.1) and therefore it has the time inversion property.

If we let Z be as in the statement of the lemma, then the function ρ^Z in the exponential part of the semigroup density must take the form (3.21).

Since A is positive definite, it can be decomposed into its eigen decomposition where each matrix is invertible and because it is symmetric all its eigenvalues are positive. That is, if $\sqrt{\Lambda}$ is a matrix containing the square roots of its positive eigenvalues on the diagonal and zeros

elsewhere then the function ρ can be decomposed as

$$2\rho^Z(x) = x^T C^T \sqrt{\Lambda}^T \sqrt{\Lambda} C x$$

where C is an invertible matrix. Therefore, there exists an invertible matrix $B = \sqrt{\Lambda} C$ such that

$$2\rho^Z(x) = \|Bx\|^2.$$

Now that we have this expression for ρ^Z , we can write the semigroup density of Z as

$$p_t^Z(x, y) = \Phi^Z\left(x, \frac{y}{t}\right) \frac{\theta^Z(y)}{t^{\frac{n+2\nu+1}{2}}} \exp\left(-\frac{\|Bx\|^2 + \|By\|^2}{2t}\right),$$

where Φ^Z and θ^Z denote the functions Φ and θ specific to the semigroup density of Z . However, more importantly, we would like to find the expression for the semigroup density of $X = B \cdot Z$. Since B is invertible, we know that X retains the conservative and Markov properties and therefore

$$\mathbb{P}_x(X_t \in dy) = \mathbb{P}_{B^{-1}x}(B \cdot Z_t \in dy) = p_t^Z(B^{-1}x, B^{-1}y) J(B) dy.$$

where $J(B)$ is the Jacobian of B . Thus, X still satisfies (H1-3') and has the semigroup density

$$p_t^X(x, y) = \Phi^Z\left(B^{-1}x, \frac{B^{-1}y}{t}\right) \frac{\theta^Z(B^{-1}y)}{t^{\frac{n+2\nu+1}{2}}} \exp\left(-\frac{\|x\|^2 + \|y\|^2}{2t}\right).$$

Moreover, by the properties of matrices, $\Phi^Z(B^{-1}\cdot, B^{-1}\cdot)$ and $\theta(B^{-1}\cdot)$ satisfy the properties of Gallardo and Yor [2005] given by (2.3) and (2.4) and so X is a process with a semigroup density of the form (3.1) satisfying (H1-4'). Therefore, it has the time inversion property and has $\rho^X(z) = \frac{\|z\|^2}{2}$.

Furthermore, to prove the second part, we also note from the above that

$$2\rho^Z(Z_t) = \|BZ_t\|^2 = \|BB^{-1} \cdot X_t\|^2 = \|X_t\|^2$$

and the equivalence in law to a squared Bessel process follows from Theorem 29. □

Remark 33. *It can be seen that B is the combination of a stretching eigenvalue matrix $\sqrt{\Lambda}$ and a matrix that represents the combination of a rotation and a reflection. We know that the constant in ρ in each dimension represents the time-scaling in that dimension and we can only jump to points of equal ρ -value. Thus, this matrix transformation takes us from a process X that is equally time-scaled in each direction to a process that is time-scaled in each direction according to $\sqrt{\Lambda}$ and then rotated or reflected. Considering the two-dimensional case more*

carefully, whereas X can jump from a point to any point on the same circle that is an equal distance from the origin, after the matrix transformation this circle is stretched, rotated and reflected so that the possible jumping destinations of Z lie on an ellipse.

Thus, in two-dimensions, the stretch matrix $\sqrt{\Lambda}$ is given by the following

$$\sqrt{\Lambda} = \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix}$$

where σ_i^2 represents the time-scaling of the process in each dimension and the matrix C is given by

$$\underline{R}_\theta = \begin{pmatrix} \cos \theta & \pm \sin \theta \\ \mp \sin \theta & \cos \theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sin \theta & \pm \cos \theta \\ \mp \cos \theta & \sin \theta \end{pmatrix}.$$

Furthermore, we can in fact see that any process with the time inversion property multiplied by a real invertible matrix gives rise to another process with the time inversion property.

Proposition 34. *If R is a Markov process on \mathbb{R}^n that enjoys the time inversion property and M is an invertible matrix then $V = M \cdot R$ is also a Markov process that enjoys the time inversion property.*

Proof. If R is a Markov process with the time inversion property, then for any Borel function f we can determine the distribution of the time homogeneous Markov process $Y^{(x)} := (Y_t^{(x)})_{t>0}$ as (1.3) for all $t, h > 0$.

Since M is an invertible matrix we know that we can define the Markov kernel of Z in terms of the Markov kernel of R

$$p_t^Z(x, A) = p_t^R(Mx, M(A))$$

and therefore, for any time homogeneous Markov process, its matrix multiplication is also a time homogeneous Markov process provided that the matrix is invertible. Thus, if we look at the time inversion of V , which we denote as $U^{(x)}$ up to an h -transform, for any Borel function f and $t, h > 0$

$$\begin{aligned} E_{a,t} [f(U_h^{(x)})] &= \mathbb{E}_x \left[f \left((t+h) V_{\frac{1}{t+h}} \right) \mid t V_{\frac{1}{t}} = a \right] \\ &= \mathbb{E}_{M^{-1}x} \left[f \left((t+h) M \cdot R_{\frac{1}{t+h}} \right) \mid t M \cdot R_{\frac{1}{t}} = a \right] \\ &= \mathbb{E}_{M^{-1}x} \left[f \left(M \cdot (t+h) R_{\frac{1}{t+h}} \right) \mid t \cdot R_{\frac{1}{t}} = M^{-1}a \right] \\ &= \mathbb{E}_{M^{-1}a} \left[f(M \cdot Y_h^{(M^{-1}x)}) \right] \end{aligned}$$

and since $Y_t^{(M^{-1}x)}$ is Markov and time homogeneous then, since M is invertible, $U_t^{(x)}$ is also Markov and time homogeneous.

□

3.6.2 Examples

To the best of our knowledge, the only known processes satisfying the time inversion property on \mathbb{R}^n are Brownian motion, Wishart processes and various extensions of the Bessel and Dunkl processes. From these, we consider two simple examples that illustrate the properties of the process $\rho(R.)$ for a Feller process R on \mathbb{R}^n that enjoys the time inversion property.

Example 35. *We consider the case of a two-dimensional Brownian motion, which we know to satisfy the time inversion property, see Shiga and Watanabe [1973]. The semigroup density of this process, which we call W , is given by [Revuz and Yor, 2005, Chapter I] for $x, y \in \mathbb{R}^n$ and thus, $\rho(z) = \frac{\|z\|^2}{2}$ and therefore, $\sqrt{2\rho(W_t)} = \|W_t\|$. We would like to show that this has the Markov semigroup density of a Bessel process.*

In order to do this, we take an arbitrary Borel set $A = (a, b) \subset \mathbb{R}^+$ and we know that

$$\mathbb{P}_x(\|W_t\| \in A) = \int_{a < \sqrt{y_1^2 + y_2^2} < b} \frac{1}{(2\pi t)} \exp\left(-\frac{\|x - y\|^2}{t}\right) dy.$$

Changing this from Cartesian to polar coordinates with $x_1 = r_x \cos \theta_x, x_2 = r_x \sin \theta_x$

$$\begin{aligned} \mathbb{P}_x(\|W_t\| \in A) &= \int_{a < \sqrt{y_1^2 + y_2^2} < b} \frac{1}{(2\pi t)} \exp\left(-\frac{\|x - y\|^2}{t}\right) dy \\ &= \int_a^b \frac{1}{(2\pi t)} \exp\left(-\frac{\|r_x\|^2 - \|r_y\|^2}{2t}\right) \int_{0+\theta_x}^{2\pi+\theta_x} \exp\left(\frac{r_x r_y \cos(\theta_x - \theta_y)}{t}\right) r_y d\theta_y dr_y \end{aligned}$$

where we have used the double-angle formula. By the identity $I_\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \theta_y} e^{\nu \theta_y} d\theta_y$, see [Abramowitz and Stegun, 1972, Chapter 9 Equation 9.6.16] and the general form in [DLMF, Eq 10.23.3],

$$\begin{aligned} \mathbb{P}_x(\|W_t\| \in A) &= \int_a^b \frac{1}{(2\pi t)} \exp\left(-\frac{\|r_x\|^2 - \|r_y\|^2}{2t}\right) \int_0^{2\pi} \exp\left(\frac{r_x r_y \cos(\theta_y)}{t}\right) r_y d\theta_y dr_y \\ &= \int_A \frac{1}{t} \exp\left(-\frac{\|r_x\|^2 - \|r_y\|^2}{2t}\right) I_0\left(\frac{r_x r_y}{t}\right) r_y dr_y, \end{aligned}$$

which is the semigroup of a two-dimensional Bessel process and only depends on r_x not θ_x . Thus, it is a Markov process in radial position.

Example 36 (Wishart Processes). *The Wishart process, see Bru [1991], is a Markov process with continuous paths that takes values on the set of real-valued $p \times p$ matrices and is given by*

$$X_t = N_t^T N_t, \tag{3.23}$$

where N_t is a process on the set of $n \times n$ -dimensional matrices whose components are Brownian motions. Bru [1991] has also shown that X_t solves the stochastic differential equation

$$dX_t = dN_t^T N_t + N_t^T dN_t + nI_p dt$$

where I_p is a p -dimensional identity matrix. By Lawi [2008] it is a process that satisfies the time inversion property on $\mathbb{R}^{n \times n}$. Importantly, its semigroup density can be written as

$$p_t(x, y) = \frac{1}{(2t)^{\frac{p(p+1)}{2}}} \mathbb{I}_{\frac{n-p-1}{4}} \left(\frac{xy}{4t^2} \right) \left(\frac{\det(y)}{\det(x)} \right)^{\frac{n-p-1}{4}} \exp \left(\frac{-1}{2t} \text{Tr}(x + y) \right).$$

Here, \det is determinant and Tr is trace. We can see from this semigroup that $\rho(x) = \frac{\text{Tr}(x)}{2}$ and, using Itô's Lemma, Bru [1991] has also shown that this is a squared Bessel process therefore

$$2\rho(X_t) = 2 \frac{\text{Tr}(X_t)}{2} \sim Q_t^{\frac{np-2}{2}},$$

where $Q_t^{\frac{np-2}{2}}$ is a squared Bessel process of index $\frac{np-2}{2}$ or dimension np .

3.6.3 Characterising the Class \mathcal{U}

In this subsection we focus on the restricted class \mathcal{U} . We begin by characterising the generators of processes within this class, before showing that, under fairly loose restrictions, it has the skew product representation. This provides a nice link to the subject of the following chapter.

We use the results of Theorem 31 to characterise this class of processes through the generator.

Corollary 37. *If X is a Feller process belonging to the set \mathcal{U} , then it can be characterised by its infinitesimal generator applied to a function $f \in C_0^2(\mathbb{R}^n \setminus \{0\})$:*

$$\mathcal{A}_X f(x) = \sum_{i=1}^n \frac{\mu_i}{x_i} \frac{\partial f}{\partial x_i} + \sum_{i=1}^n \frac{1}{2} \frac{\partial^2 f}{\partial x_i^2} + \int (f(y) - f(x) - l_\epsilon(y - x) \cdot \nabla f(x)) n(x, dy) \quad (3.24)$$

where $\sum_{i=1}^n \mu_i = 2\nu + 2 - n$. Here ν is the index of the Bessel process whose distribution is given by $\hat{\rho}(X_t)$.

Proof. In this case we can use Theorem 31 to determine \mathcal{U} through the infinitesimal generator. Since we know

$$\frac{\partial \hat{\rho}}{\partial x_i} = \frac{x_i}{\|x\|}, \quad \frac{\partial^2 \hat{\rho}}{\partial x_i^2} = \frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} \text{ and } \frac{\partial^2 \hat{\rho}}{\partial x_i \partial x_j} = \frac{x_i x_j}{\|x\|^3},$$

we can use the results in Theorem 31 to conclude that the Gaussian part of the generator is simply the identity matrix. Furthermore, by (3.20) we can see that

$$\frac{2\nu + 1}{2\|x\|} = \sum_{i=1}^n \mu_i(x) \frac{x_i}{\|x\|} + \sum_{i,j=1}^n \frac{\sigma_{ij}}{2} \left(\frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} \right)$$

so $\mu_i(x) = \frac{\mu_i}{x_i}$ for some $\sum_{i=1}^n \mu_i = 2\nu + 2 - n$

Therefore, by the above, if $\rho^X(x) = \sum_{i=1}^n \frac{\|x\|^2}{2}$ then the infinitesimal generator of the process can be written as in the statement of the corollary.

□

This provides us with a form of the generator, and therefore a characterisation, of \mathcal{U} . However, we are also interested in the other direction, namely, do processes with an infinitesimal generator of the form given in (3.24) necessarily have the time inversion property? We answer this in the next chapter through the skew product structure of processes that relies on describing processes in terms of their radial and angular parts, see Section 1.6. We discuss the skew product structure for processes in \mathcal{U} here.

As alluded to above, the skew product representation is integral to proving sufficient conditions that guarantee the time inversion property for Feller processes and it forms the foundation for the work in our final chapter. In this section we first define the skew product representation before focussing on the link between it and processes in \mathcal{U} .

Definition 38. *If X is a Feller process that can be written $X_t = r_t \Theta_{H_t}$, where r is a, possibly time-scaled, Bessel process; H_t is the time-change given by $H_t = \int_0^t r_u^{-2} du$ and $\Theta := (\Theta_t)_{t \geq 0}$ is a Feller process on S^{n-1} independent of r then we say X has the skew product representation.*

This skew product representation gives us another way of characterising \mathcal{U} under some restrictions.

Proposition 39. *If X is a member of \mathcal{U} and $\hat{\Phi}(x, y) = \Phi(x, y)\theta(\hat{y})$ in the form of its semigroup density (2.2) is isotropically invariant, then X has the skew product representation.*

In the above we take isotropically invariant to mean that $\hat{\Phi}(T(x), y) = \hat{\Phi}(x, T^{-1}(y))$ for any $T \in O(n)$

Proof. This proof hinges on a result from Liao and Wang [2011], which shows that any isotropic self-similar Markov process can be written in a generalised skew product structure. Thus, we are only required to show that X is isotropic and that the skew product structure of X reduces to our skew product form given in Definition 38.

The 2-self-similarity of the process follows from the form of the semigroup (3.1), so we immediately turn to showing that a member of the isotropic class is indeed isotropic. That is, if X is a member of the isotropic class with a semigroup density of the form $p_t^X(x, y)$, then

$$p_t^X(T(x), y) = p_t^X(x, T^{-1}(y))$$

for any $T \in O(n)$, the set of orthogonal matrices on \mathbb{R}^n . This follows since $\hat{\Phi}$ is isotropic by the conditions of the proposition and the rest of the semigroup density does not depend on the angular parts of its arguments because $\rho^X(x) = \frac{\|x\|^2}{2}$.

Since X is isotropic and the radial part has continuous paths almost surely by Theorem 29, we have that it can be written in the skew product structure as in the statement of the proposition.

However, by self-similarity and the almost surely continuous paths of the radial part of the process (Theorem 26) r is a, possibly time-scaled, Bessel process and $h(x) = Kx^{-2}$ for some constant $K > 0$.

□

Remark 40. *Since \mathcal{V} can be obtained from \mathcal{U} by multiplication by an invertible matrix, we can also use the skew product representation and multiplication by a matrix to obtain a characterisation of \mathcal{V} similar to the characterisation of \mathcal{U} seen here.*

Chapter 4

The Time Inversion Property and the Skew Product Representation on \mathbb{R}^n

4.1 Introduction

The previous chapter laid down a set of restrictions that processes enjoying the time inversion property, see Definition 4, are required to satisfy, however, this characterisation does not accommodate easy or algorithmic construction of new examples. We address this problem here.

In this chapter, we focus on processes that can be expressed through a skew product representation as discussed in Section 1.6, which we define in detail in Definition 41. As a main result, we prove that processes of this form enjoy the time inversion property. From this standpoint, we can use variations of the skew product representation to generate new examples of processes with the time inversion property. This is aided by the flexible construction of the skew product.

To our knowledge, the only examples of processes with the time inversion property can largely be seen as the result of the Gallardo and Yor [2005] and Lawi [2008] semigroup density. Thus, this skew product construction provides a technique for finding processes enjoying the time inversion property where the absolute continuity of the semigroup with respect to the Lebesgue measure is not required.

The importance of the skew product representation to self-similar processes on \mathbb{R}^n for $n \geq 2$ was first recognised by Graversen and Vuolle-Apiala [1986b], who worked on extending Lamperti's idea of self-similar Markov processes restricted to the positive case. Provided that the process was invariant under $O(n)$ - the set of orthogonal matrices, Graversen and Vuolle-Apiala [1986b] were able to show that a self-similar process could be written in the form of a skew product representation, where the self-similarity stemmed from the radial part of the process. We note here that they did not restrict the radial part to only Bessel processes as we do in the sequel.

Furthermore, Vuolle-Apiala [1992] then used this expression for a self-similar process in \mathbb{R}^n to deal with excursions and the related recurrent extensions of the radial part of the skew product representation. This is analogous to the recurrent extensions of the Bessel process for $\nu < 0$ discussed by Rivero [2005] and in our form of the skew product representation.

Other than the convenience of the skew product representation when characterising self-similar processes in higher dimensions our motivation behind using it to characterise processes with the time inversion property is two-fold.

Firstly, we have already seen on \mathbb{R}^+ that the only examples of processes with the time inversion property are Bessel processes in the wide sense, which suggests that some part of any process with the time inversion property must be associated with the Bessel process, under a possible time-scaling, in some way. Furthermore, on \mathbb{R} our foremost examples of processes with the time inversion property are the generalised Dunkl . Moreover, it has been shown by Gallardo and Yor [2006] that, when the drift parameter ν is greater than zero, the generalised Dunkl process $D_t^{(\nu, \lambda)}$ can be written in an unorthodox skew product structure as

$$D_t^{(\nu, \lambda)} = r_t e^{i\pi N_{H_t}^\lambda},$$

where r is a Bessel process with index $\nu \geq 0$ and $N_t^{(\lambda)}$ is a Poisson process with parameter $\lambda \geq 0$ under the time change

$$H_t = \int_0^t \frac{1}{r_u^2} du. \quad (4.1)$$

This suggests that there is some link between the skew product representation and the time inversion property.

Secondly, a lot of the literature in the area also prompts this link. For example, Vuolle-Apiala [2012] was able to show that a 2-self-similar diffusion process on \mathbb{R}^n for $n \geq 2$ had the time inversion property provided that it was rotation invariant (RI). This was proved using the skew product representation and the finite dimensional distributions of the process under rotation invariance.

In the workings of this chapter we also see many connections with the Hartman and Watson [1974] distribution, which is the distribution of the time change H_t in (4.1), initiated at zero and conditional on the current position of the Bessel process in the time change. Namely, it is the distribution given by

$$\mathbb{P}_{r_0}(H_t \in ds | r_t = r).$$

Much work has been conducted in the area around this distribution and Pitman and Yor [1981] even managed to determine the distribution through the Laplace transform

$$\mathbb{E}_{r_0} \left[e^{-\lambda H_t} | r_t = r \right] = \frac{I_{\sqrt{2\lambda + \nu^2}} \left(\frac{rr_0}{t} \right)}{I_\nu \left(\frac{rr_0}{t} \right)} \quad (4.2)$$

for $r, r_0, \lambda > 0$, where $\nu > -1$ is the index of the associated Bessel process. For more information on this distribution we refer the reader to Pitman and Yor [1981] and [Revuz and Yor, 2005, Chapter VII Proposition VIII.3.8].

This chapter looks at a similar approach to that considered by Vuolle-Apiala [2012], but in a way that allows us to consider processes outside the class of diffusions. In doing so, our scope is also extended beyond the rotational invariance restriction.

The remainder of the chapter is as follows. In Section 4.2, we define the skew product representation and review some preliminaries related to the discussion in the sequel. Section 4.3 contains the main result of the chapter. Here, we show that processes with the skew product representation necessarily enjoy the time inversion property and prove that the known examples on \mathbb{R} fall into this category, up to an h -transform. Section 4.4 utilises this tool in the two-dimensional case. We show that the approach of Vuolle-Apiala [2012] is contained within that considered by Lawi [2008] by explicitly determining the semigroup density of this class of processes. From this, we then construct several examples of processes with the time inversion property. Finally, we show that our extension of Vuolle-Apiala [2012]’s method goes beyond the scope of Lawi [2008]. Namely, we use it to construct a process with the time inversion property that does not satisfy the absolute continuity restriction.

4.2 Construction of the Skew Process, Assumptions and Notation

In this section, we consider the assumptions and notation of the skew product representation as it is used in the sequel and focus on some special cases of the skew product representation, particularly in two-dimensions. We also take this opportunity to define the skew product representation itself in terms of restrictions on the angular and radial parts.

Definition 41. *Let R be a Feller process on \mathbb{R}^n that can be written as $R_t = r_t \Theta_{H_t}$. Here, r is a, possibly time-scaled, Bessel process initiated at $r_0 > 0$ with index $\nu \geq 0$; H_t is the continuous and strictly increasing time-change given by (4.1) and $\Theta := (\Theta_t)_{t \geq 0}$ is a time-homogeneous Feller process on S^{n-1} which is independent of r . If these conditions hold, then we say R has the skew product representation.*

We now move to defining the notation of the chapter. We use the definition of the time inversion property given in Definition 4 and let $(R_t, \mathbb{P}_x)_{t \geq 0, x \in \mathbb{R}^n}$ be a self-similar Markov process on \mathbb{R}^n , which has a skew product representation as defined in Definition 41. We also note that this can be expressed as $R_t = r_t g(\underline{\gamma}_{H_t})$ for a function g representing spherical coordinates, see

Section 3.2, as follows:

$$r_t g(\underline{\gamma}_{H_t}) = r_t \begin{pmatrix} \cos(\gamma_{H_t}^{(1)}) \\ \sin(\gamma_{H_t}^{(1)}) \cos(\gamma_{H_t}^{(2)}) \\ \vdots \\ \sin(\gamma_{H_t}^{(1)}) \dots \sin(\gamma_{H_t}^{(n-2)}) \cos(\gamma_{H_t}^{(n-1)}) \\ \sin(\gamma_{H_t}^{(1)}) \dots \sin(\gamma_{H_t}^{(n-2)}) \sin(\gamma_{H_t}^{(n-1)}) \end{pmatrix}, \quad (4.3)$$

where we make the additional assumption that $\underline{\gamma} := (\gamma_t^{(1)} \dots \gamma_t^{(n-1)})_{t \geq 0}$ are $(n-1)$ time-homogeneous Feller processes each taking values on $[0, 2\pi) \times [0, \pi)^{n-2}$ that are independent of the, possibly time-scaled, Bessel process $r = (r_t)_{t \geq 0}$ with index $\nu \geq 0$ on \mathbb{R}^+ . The components of the process can be taken to be on the real line modulo $[0, 2\pi)$ if processes on \mathbb{R} are taken. The time-change itself is a function of the Bessel process and therefore, by the independence of the Bessel process, this time-change is also independent of the angular part of the process $\underline{\gamma}$. The positivity of the index of the Bessel process also implies that it avoids zero almost surely meaning that the time change remains finite in finite time. As denoted in the probability measure, we also initiate the n -dimensional process R at a point $\underline{x} \in \mathbb{R}^n$ where $(r_0, \underline{\gamma}_0)$ are the spherical coordinates of the point \underline{x} and $\underline{\gamma}_0 := (\gamma_0^{(1)} \dots \gamma_0^{(n-1)})$. Thus, we can see that r and $\underline{\gamma}$ are initiated at r_0 and $\underline{\gamma}_0$ respectively.

Finally, in this chapter we also use the conditional distributions of each of the Feller processes $\gamma_{H_t}^{(i)}$ with respect to the Bessel process r , which we discuss in the following remark.

Remark 42. We define the conditional distributions for a process $\underline{\gamma}_{H_t}$ conditioned on a, possibly time-scaled Bessel process, r as

$$\mathbb{P}_x(\underline{\gamma}_{H_t} \in d\underline{\phi} | r_t = r_0) = \frac{\mathbb{P}_x(\underline{\gamma}_{H_t} \in d\underline{\phi}; r_t \in dr_0)}{\mathbb{P}_x(r_t \in dr_0)}$$

for $r_0, t > 0$, $x \in \mathbb{R}^n$ and $\underline{\phi} \in S^{n-1}$, where we comment that the denominator is always strictly positive in this case. We also note that this is equivalent to the definition given in [Revuz and Yor, 2005, Chapter XI Definition XI.3.1] as the distribution given by the almost surely unique function μ that satisfies

$$\mathbb{P}_x(\underline{\gamma}_{H_t} \in A) = \int \mu(x, r_0, t; A) \mathbb{P}_x(r_t \in dr_0)$$

for all $x \in \mathbb{R}^n$, $r_0, t > 0$ and A a Borel subset of $S^{n-1} \subset \mathbb{R}^n$.

We also observe the similarity between this and the Hartman and Watson [1974] distribution, which arises in the specific case when $\gamma_t = t$ on \mathbb{R}^2

These conditional distributions always exist away from the origin by the positivity of the semigroup of the Bessel process and so for all $x > 0$ by the restriction on ν and [Revuz and Yor, 2005, Chapter XI Proposition XI.1.5]. Furthermore, we also take this opportunity to note

that as the function g in (4.3) is both bijective and invertible, by the properties of spherical coordinates, the Markov property of the process in \mathbb{R}^n follows from the Markov property of the component processes.

We also remark that when $n = 2$ (4.3) reduces to

$$R_t = r_t \begin{pmatrix} \cos(\gamma_{H_t}) \\ \sin(\gamma_{H_t}) \end{pmatrix}, \text{ or equivalently, } r_t e^{i\gamma_{H_t}}, \quad (4.4)$$

where we have exchanged the real plane for the complex plane. In its most simple case, this skew product in two dimensions can be seen to reduce to standard two-dimensional Brownian motion when r is the Bessel process with index $\nu = 0$ and γ is an independent Brownian motion, see Le Gall [1992].

In two dimensions, the independence of the radial and time-changed angular parts permit us to write the infinitesimal generator as the sum of the two generators applied to an $f(r, \gamma) \in C_0^2 \times \mathcal{D}_{\mathcal{A}_\gamma}$

$$\mathcal{A}_R f(x) = \mathcal{A}_r f(r_0) + \frac{1}{r_0^2} \mathcal{A}_\gamma f(\gamma_0),$$

where $\mathcal{D}_{\mathcal{A}_\gamma}$ is the domain of the process γ . In addition, \mathcal{A}_r is the generator of a time-scaled Bessel process and when γ is Feller and time homogeneous

$$\mathcal{A}_\gamma f(\gamma_0) = \mu(\gamma_0) \frac{\partial f}{\partial \gamma_0} + \sigma(\gamma_0) \frac{\partial^2 f}{\partial \gamma_0^2} + \int (f(\phi) - f(\gamma_0) - l_\epsilon(\phi - \gamma_0) \cdot \nabla f(\gamma_0)) n(\gamma_0, d\phi).$$

This is obtained using Volkonski's Theorem, see Volkonski [1958].

4.3 The Time Inversion Property for the Skew Product in n Dimensions

In this section, we prove the main result of the chapter. Namely, we show that the existence of an expression for a process as a skew product representation, as defined in Definition 41, is sufficient to invoke the time inversion property. As an example, we then consider the processes satisfying Lawi's restrictions on \mathbb{R} as a subset of this class, up to an h -transform and time-scaling.

In order to show the time inversion property for the n -dimensional process given by (4.3), we would like to show that for any x in the state space S of R , the inverted process given by $Y_t^{(x)} := tR_{\frac{1}{t}} \in \mathbb{R}^n$ as defined in Gallardo and Yor [2005] is time homogeneous. That is, its distribution $q_{t,t+h}^{(x)}(a, b)$ is independent of time t . On account of the expression for the distribution of this process (and therefore the inverted process up to an h -transform) given in

(1.3), this is equivalent to showing that the expectation

$$\mathbb{E}_{a,t} \left[f(Y_{t+h}^{(x)}) \right] = \mathbb{E}_x \left[f \left((t+h)r_{\frac{1}{t+h}} g \left(\gamma_{H_{\frac{1}{t+h}}} \right) \right) \middle| tr_{\frac{1}{t}} g \left(\gamma_{H_{\frac{1}{t}}} \right) = a \right] \quad (4.5)$$

for Y initiated at time t and spatial point a is independent of t for any Borel f and $h > 0$, where we have used the function $r_t g(\gamma_t)$ to represent R in skew product form. Hence, showing the time homogeneity of $Y_t^{(x)}$ for a particular $x \in S$ amounts to showing no dependence upon t in the conditional measure

$$\mathbb{P}_x \left((t+h)r_{\frac{1}{t+h}} \in d\mathbf{r}; \gamma_{H_{\frac{1}{t+h}}} \in d\phi \middle| tr_{\frac{1}{t}} = r_a; \gamma_{H_{\frac{1}{t}}} \in d\theta_a \right),$$

where r_a and θ_a are the radial and angular parts of a respectively.

Our theorem, and main result of this section, hinges in great measure upon the following lemma. It uses a generalisation of the Hartman-Watson distribution to show the independence of the conditioned time change upon the variable t that represents the starting time of the inverted process.

Lemma 43. *Let $\underline{\gamma} := ((\gamma_t^{(1)} \dots \gamma_t^{(n)})^T)_{t \geq 0}$ be independent Feller processes with conditional distributions given by Definition 42 and let H_t be the independent time change given by (4.1), where r is a, possibly time-scaled, Bessel process of index $\nu \geq 0$ initiated at $r_0 > 0$. Then the conditioned probability measure*

$$\mathbb{P}_{r_0, \underline{\gamma}_0} \left(\gamma_{H_{\frac{1}{t}}} \in dy \middle| r_{\frac{1}{t}} = \frac{r_a}{t} \right), \quad (4.6)$$

which represents the distribution of the angular parts of the skew process, is independent of $t > 0$.

Furthermore, in a similar way, the probability measure

$$\mathbb{P}_{\frac{r_0}{t+h}, \underline{\gamma}_0} \left(\gamma_{H_{\frac{1}{t} - \frac{1}{t+h}}} \in dy \middle| r_{\frac{1}{t} - \frac{1}{t+h}} = \frac{r_a}{t} \right), \quad (4.7)$$

where $\mathbb{P}_{\frac{r}{t+h}, \underline{\gamma}_0}$ represents the measure of the conditioned process in which r and $\underline{\gamma}$ are initiated at $\frac{r}{t+h}$ and $\underline{\gamma}_0$ respectively, is also independent of t and only depends on $h > 0$ for all angular components $\underline{\gamma}_0, y \in \mathbb{R}^{n-1}$ and $r_a, r_0 > 0$.

Proof. To prove the result we show that the probability distribution (4.6) is independent of the variable $t > 0$ for all $r_0, r_a > 0$ and $\phi, \underline{\gamma}_0 \in S^{n-1}$, this is equivalent to showing that

$$\mathbb{P}_{r_0, \underline{\gamma}_0} \left(\gamma_{H_{\frac{1}{t}}} \in d\phi \middle| r_{\frac{1}{t}} = \frac{r_a}{t} \right) = \mathbb{P}_{r_0, \underline{\gamma}_0} \left(\gamma_{H_{\frac{1}{s}}} \in d\phi \middle| r_{\frac{1}{s}} = \frac{r_a}{s} \right) \quad (4.8)$$

for all $s > 0$.

Initially, we focus upon just the stochastic time change H_t . Thus, we show the independence of t in the conditioned measure

$$\mathbb{P}_{r_0}^t := \mathbb{P}_{r_0} \left(H_{\frac{1}{t}} \in ds \middle| r_{\frac{1}{t}} = \frac{r_a}{t} \right).$$

We do this by showing that the Laplace transforms of the measures are equal for any value of t . The result will follow because Laplace transforms of measures are injective by Lerch's Theorem.

Taking the Laplace transform in the case of a fixed $t > 0$ and for a $\lambda > 0$ we obtain

$$\mathcal{L}(\mathbb{P}_{r_0}^t)(\lambda) = \int_0^\infty e^{-\lambda s} \mathbb{P}_{r_0} \left(H_{\frac{1}{t}} \in ds \middle| r_{\frac{1}{t}} = \frac{r_a}{t} \right) = \mathbb{E}_{r_0} \left[e^{-\lambda H_{\frac{1}{t}}} \middle| r_{\frac{1}{t}} = \frac{r_a}{t} \right].$$

Perceiving this as the Laplace transform of the Hartman-Watson distribution, the expression is given in terms of modified Bessel functions in (4.2), courtesy of Pitman and Yor [1981], and using this expression in our case, where we have used $\nu \geq 0$ for the index of the Bessel process, gives

$$\mathcal{L}(\mathbb{P}_{r_0}^t)(\lambda) = \mathbb{E}_{r_0} \left[e^{-\lambda H_{\frac{1}{t}}} \middle| r_{\frac{1}{t}} = \frac{r_a}{t} \right] = \frac{I_{\sqrt{2\lambda+\nu^2}}(r_0 r_a)}{I_\nu(r_0 r_a)}.$$

This is independent of time for all $\lambda > 0$ and therefore, also for any $\lambda > 0$,

$$\mathcal{L}(\mathbb{P}_x^t)(\lambda) = \mathcal{L}(\mathbb{P}_x^s)(\lambda) \tag{4.9}$$

and so the measures are equivalent almost everywhere. Moreover, this is also independent of any possible time-scaling of the Bessel process provided $\sigma > 0$.

In a similar way, if we take the same Laplace transform for the measure

$$\mathbb{P}_{\frac{r_0}{t+h}} \left(H_{\frac{1}{t}-\frac{1}{t+h}} \in ds \middle| r_{\frac{1}{t}-\frac{1}{t+h}} = \frac{r_a}{t} \right)$$

for any $r_0, t, h, r_a > 0$, then by using the same method as above

$$\mathbb{E}_{\frac{r_0}{t+h}} \left[e^{-\lambda H_{\frac{1}{t}-\frac{1}{t+h}}} \middle| r_{\frac{1}{t}-\frac{1}{t+h}} = \frac{r_a}{t} \right] = \frac{I_{\sqrt{2\lambda+\nu^2}}\left(\frac{r_0 r_a}{h}\right)}{I_\nu\left(\frac{r_0 r_a}{h}\right)}, \tag{4.10}$$

which is also independent of t and only depends on the difference between the time variables given by h .

Now that we have the independence of the conditioned time-changed measure, we can use subordination and the law of total probability to show the full lemma. We show that for any $t > 0$ and any Borel function f with compact support

$$\mathbb{E}_{r_0, \gamma_0} \left[f(\gamma_{H_{\frac{1}{t}}}) \middle| r_{\frac{1}{t}} = \frac{r_a}{t} \right]$$

is independent of t and this is equivalent to showing it for the measure (4.8) almost surely. We

also note that finiteness is guaranteed for $t < \infty$ by the compact support of f . Thus, using the law of total probability and the independence of $\underline{\gamma}$ with respect to the additive time change H_t ,

$$\mathbb{E}_{r_0, \underline{\gamma}_0} \left[f(\underline{\gamma}_{H_{\frac{1}{t}}}) | r_{\frac{1}{t}} = \frac{r_a}{t} \right] = \int_0^\infty \mathbb{E}_{r_0, \underline{\gamma}_0} [f(\underline{\gamma}_s)] \mathbb{P}_{r_0} \left(H_{\frac{1}{t}} \in ds | r_{\frac{1}{t}} = \frac{r_a}{t} \right).$$

From (4.9) almost everywhere, and thus the equality of these measures given by the injective property of Laplace transforms, we know that the measure is independent of t and therefore $\mathbb{P}_{r_0} \left(H_{\frac{1}{t}} \in ds | r_{\frac{1}{t}} = \frac{r_a}{t} \right) = \mu(r_0, r_a; ds)$ for some measure μ . Substituting this measure, which has no dependence on t , into our expression yields

$$\mathbb{E}_{r_0, \underline{\gamma}_0} \left[f(\underline{\gamma}_{H_{\frac{1}{t}}}) | r_{\frac{1}{t}} = \frac{r_a}{t} \right] = \int_0^\infty \mathbb{E}_{r_0, \underline{\gamma}_0} [f(\underline{\gamma}_s)] \mu(r_0, r_a; ds),$$

which is bounded by the properties of a sub-Markov probability measure and the compact support of f . This has no dependence on t and therefore, the measure in (4.8) has no dependence on t and we have our result.

Performing the same substitutions in the second case and using (4.10) we only have dependence on h in equation (4.7). □

Equipped with this lemma, we are now in a position to prove the main theorem of this section and show that any process, for which there exists a skew product representation of the form given in Definition 41 and consequently (4.3), has the time inversion property on \mathbb{R}^n .

Theorem 44. *If R is a process on \mathbb{R}^n that has the skew product representation given by Definition 41, then it enjoys the time inversion property.*

In this proof we are required to show the time homogeneity of the inverted process. Lemma 43 has already shown that the inversion of the angular part is time homogeneous conditional on the position of the radial part and the property is already well known for the Bessel process, see Watanabe [1975]. We combine these two properties to prove the result for the skew product.

Proof. We first note that the skew product representation given in Definition 41 is equivalent to the skew product representation expressed in n -spherical coordinates by (4.3) and so we use the spherical coordinates representation in this proof.

By Gallardo and Yor [2005], we can express the distribution of the time inverted process $Y_t^{(x)}$, unique up to an h -transform, of a Feller process R as the conditional distribution (1.3) for any Borel f with compact support. It has already been shown (see Gallardo and Yor [2005]) that $Y_t^{(x)}$ defined in this way is a Markov process, although it is time inhomogeneous in general. Thus, to show the time inversion property we show the time homogeneity of this process.

As discussed previously, employing (1.3) as a representation of the distribution of $Y^{(x)}$, which gives the distribution of the inverted process up to an h -transform, showing the time

inversion property is equivalent to showing the independence of t in (4.5) for all Borel f . Thus, if we can show

$$\mathbb{E}_x \left[f \left((t+h) R_{\frac{1}{t+h}} \right) | t R_{\frac{1}{t}} = a \right] = \mathbb{E}_x \left[f \left((s+h) R_{\frac{1}{s+h}} \right) | s R_{\frac{1}{s}} = a \right]$$

for the skew product $R_t = r_t g(\underline{\gamma}_t)$ for all $s, t > 0$, then we are done.

Considering this expectation and using the skew product representation of R given in (4.3), the process can be expressed in terms of a joint probability distribution

$$\begin{aligned} & \mathbb{E}_x \left[f \left((t+h) g \left(r_{\frac{1}{t+h}}, \underline{\gamma}_{H_{\frac{1}{t+h}}} \right) \right) | t g \left(r_{\frac{1}{t}}, \underline{\gamma}_{H_{\frac{1}{t}}} \right) = a \right] \\ &= \int \int f(g(z, \underline{\phi})) \mathbb{P}_x \left((t+h) r_{\frac{1}{t+h}} \in dz; \underline{\gamma}_{H_{\frac{1}{t+h}}} \in d\underline{\phi} \mid tr_{\frac{1}{t}} = r_a; \underline{\gamma}_{H_{\frac{1}{t}}} = \underline{\theta}_a \right) \\ &= \int \int f(g(z, \underline{\phi})) \frac{\mathbb{P}_x \left((t+h) r_{\frac{1}{t+h}} \in dz; \underline{\gamma}_{H_{\frac{1}{t+h}}} \in d\underline{\phi}; \quad tr_{\frac{1}{t}} \in dr_a; \underline{\gamma}_{H_{\frac{1}{t}}} \in d\underline{\theta}_a \right)}{\mathbb{P}_x \left(tr_{\frac{1}{t}} \in dr_a; \underline{\gamma}_{H_{\frac{1}{t}}} \in d\underline{\theta}_a \right)}, \end{aligned}$$

where we have used the existence of the regular conditional distribution (discussed in Remark 42), which is guaranteed by the positivity of the Bessel semigroup. Using Bayes' Theorem on the joint distribution and the Markov property on the result gives

$$\begin{aligned} & \mathbb{E}_x \left[f \left((t+h) g \left(r_{\frac{1}{t+h}}, \underline{\gamma}_{H_{\frac{1}{t+h}}} \right) \right) | t g \left(r_{\frac{1}{t}}, \underline{\gamma}_{H_{\frac{1}{t}}} \right) = a \right] \\ &= \int \int f(g(z, \underline{\phi})) \frac{\mathbb{P}_x \left((t+h) r_{\frac{1}{t+h}} \in dz; \quad \underline{\gamma}_{H_{\frac{1}{t+h}}} \in d\underline{\phi} \right) \mathbb{P}_{\frac{z}{t+h}, \underline{\phi}} \left(tr_{\frac{1}{t} - \frac{1}{t+h}} \in dr_a; \underline{\gamma}_{H_{\frac{1}{t} - \frac{1}{t+h}}} \in d\underline{\theta}_a \right)}{\mathbb{P}_x \left(tr_{\frac{1}{t}} \in dr_a; \underline{\gamma}_{H_{\frac{1}{t}}} \in d\underline{\theta}_a \right)}, \end{aligned}$$

where $\mathbb{P}_{\frac{z}{t+h}, \underline{\phi}}$ represents the measure of the joint distribution when the radial part is initiated at $\frac{z}{t+h}$ and the angular part is initiated at $\underline{\phi}$. Employing Bayes' Theorem again, using the angular part as the subject of the conditioning on this occasion and re-arranging to match angular and

radial parts

$$\begin{aligned}
& \mathbb{E}_x \left[f \left((t+h)g \left(r_{\frac{1}{t+h}}, \gamma_{H_{\frac{1}{t+h}}} \right) \right) \middle| tg \left(r_{\frac{1}{t}}, \gamma_{H_{\frac{1}{t}}} \right) = a \right] \\
&= \int \int f(g(z, \underline{\phi})) \frac{\mathbb{P}_{\frac{z}{t+h}, \underline{\phi}} \left(\gamma_{H_{\frac{1}{t+h}}} \in d\underline{\theta}_a \middle| tr_{\frac{h}{t(t+h)}} \in dr_a \right) \mathbb{P}_{\frac{z}{t+h}, \underline{\phi}} \left(tr_{\frac{h}{t(t+h)}} \in dr_a \right)}{\mathbb{P}_x \left(\gamma_{H_{\frac{1}{t}}} \in d\underline{\theta}_a \middle| tr_{\frac{1}{t}} = r_a \right) \mathbb{P}_x \left(tr_{\frac{1}{t}} \in dr_a \right)} \\
&\quad \times \mathbb{P}_x \left(\gamma_{H_{\frac{1}{t+h}}} \in d\underline{\phi} \middle| r_{\frac{1}{t+h}} = \frac{z}{t+h} \right) \mathbb{P}_x \left((t+h)r_{\frac{1}{t+h}} \in dz \right) \\
&= \int \int f(g(z, \underline{\phi})) \frac{\mathbb{P}_x \left(\gamma_{H_{\frac{1}{t+h}}} \in d\underline{\phi} \middle| r_{\frac{1}{t+h}} = \frac{z}{t+h} \right) \mathbb{P}_{z, \underline{\phi}} \left(\gamma_{H_{\frac{h}{t(t+h)}}} \in d\underline{\theta}_a \middle| tr_{\frac{h}{t(t+h)}} = r_a \right)}{\mathbb{P}_x \left(\gamma_{H_{\frac{1}{t}}} \in d\underline{\theta}_a \middle| tr_{\frac{1}{t}} = r_a \right)} \\
&\quad \frac{\mathbb{P}_{\frac{z}{t+h}} \left(tr_{\frac{h}{t(t+h)}} \in dr_a \right) \mathbb{P}_x \left((t+h)r_{\frac{1}{t+h}} \in dz \right)}{\mathbb{P}_x \left(tr_{\frac{1}{t}} \in dr_a \right)}.
\end{aligned}$$

By the time inversion property of the Bessel process, assured by Watanabe [1975] and Lawi [2008], we know that $\frac{\mathbb{P}_z \left(tr_{\frac{h}{t(t+h)}} \in dr_a \right) \mathbb{P}_x \left((t+h)r_{\frac{1}{t+h}} \in dz \right)}{\mathbb{P}_x \left(tr_{\frac{1}{t}} \in dr_a \right)}$ is independent of t and therefore it is equivalent to a kernel that is independent of time, which we call $K(x, h, r_a, dz)$. If we substitute this expression of the kernel that is independent of t into the above then we are left with

$$\int \int f(g(z, \underline{\phi})) \frac{\mathbb{P}_x \left(\gamma_{H_{\frac{1}{t+h}}} = \underline{\phi} \middle| r_{\frac{1}{t+h}} = \frac{z}{t+h} \right) \mathbb{P}_{z, \underline{\phi}} \left(\gamma_{H_{\frac{h}{t(t+h)}}} = \underline{\theta}_a \middle| tr_{\frac{h}{t(t+h)}} = r_a \right)}{\mathbb{P}_x \left(\gamma_{H_{\frac{1}{t}}} = \underline{\theta}_a \middle| tr_{\frac{1}{t}} = r_a \right)} K(x, h, r_a, dz).$$

Furthermore, the three conditioned measures are all independent of the variable t by Lemma 43 and so the entire expression is independent of t . This means that

$$\int \int f(g(z, \underline{\phi})) \mathbb{P}_x \left(r_{\frac{1}{t+h}} \in d\frac{z}{t+h}; \gamma_{H_{\frac{1}{t+h}}} \in d\underline{\phi} \middle| tr_{\frac{1}{t}} = r_a; \gamma_{H_{\frac{1}{t}}} = \underline{\theta}_a \right)$$

is independent of t , hence

$$\mathbb{E}_x \left[f \left((t+h)R_{\frac{1}{t+h}} \right) \middle| tR_{\frac{1}{t}} = a \right] = \mathbb{E}_x \left[f \left((s+h)R_{\frac{1}{s+h}} \right) \middle| sR_{\frac{1}{s}} = a \right]$$

for all $s, t > 0$ and we have time homogeneity for the inverted process. \square

To illustrate this result, we consider a simple example that is already covered by the

results of Vuolle-Apiala [2012].

Example 45 (The case in two dimensions where γ is a Lévy process with continuous paths). *If we take the case where γ is a Brownian motion plus drift, then its Laplace exponent ψ is finite and it is given by $\psi(\theta) = \lambda\theta + \frac{\sigma^2}{2}\theta^2$. We can now compute the skew representation with γ as the angular component by the Itô formula. Taking $X_t = r_t \cos \theta_t$, $Y_t = r_t \sin \theta_t$ and using the increments*

$$dr_t = \frac{2\nu + 1}{r_t} dt + \sigma_r dB_t, \quad d\gamma_t = \lambda dt + \sigma_\gamma d\tilde{B}_t$$

then by the Itô Formula we can compute the X component

$$\begin{aligned} dX_t &= \frac{dX}{dr} dr_t + \frac{dX^2}{dr^2} d\langle r, r \rangle_t + \frac{dX}{d\gamma} d\gamma_{H_t} + \frac{dX^2}{d\gamma^2} d\langle \gamma, \gamma \rangle_{H_t} \\ &= \left[\frac{X_t}{\sqrt{X_t^2 + Y_t^2}} \frac{2\nu + 1}{\sqrt{X_t^2 + Y_t^2}} - \frac{Y_t}{\sqrt{X_t^2 + Y_t^2}} \lambda - \frac{X_t}{X_t^2 + Y_t^2} \sigma_\gamma^2 \right] dt \\ &\quad + \frac{X_t}{\sqrt{X_t^2 + Y_t^2}} \sigma_r dB_t - \frac{Y_t}{\sqrt{X_t^2 + Y_t^2}} \sigma_\gamma d\tilde{B}_t \end{aligned}$$

and similarly for Y .

When $\sigma_r = \sigma_\gamma = 1$, the separate Gaussian parts become equivalent in law to a single Brownian motion by Lévy's Characterisation Theorem, see [Revuz and Yor, 2005, Chapter IV Theorem IV.3.6], and when $\lambda = \nu = 0$ this is a two-dimensional Brownian motion as seen in Le Gall [1992].

Furthermore, this particular skew representation has the rotational invariance property on account of the increments of γ . That is,

$$\mathbb{E}_{r_x, \theta_x} [f(r_t e^{i\gamma_{H_t}})] = \mathbb{E}_{r_x, \theta_x - \theta_y} [f(r_t e^{i(\gamma_{H_t} + \theta_y)})].$$

This is because both the time change and the Bessel process are independent of the angular part. This rotational invariance property (RI) combined with the self-similarity property implies that the skew product representation has the time inversion property by the work of Vuolle-Apiala [2012].

Theorem 44 tells us that, under some fairly minor assumptions, any process R that can be written in the form of the skew product representation also enjoys the time inversion property. Unsurprisingly, this result gives rise to many corollaries. Perhaps more surprisingly, the first of these corollaries applies to processes in only one dimension.

4.3.1 Example: Skew Product Enjoying the Time Inversion Property on \mathbb{R}

Recall that in Chapter 2 we proved that if a Feller process $(R_t)_{t \geq 0}$ on \mathbb{R} , which avoids zero almost surely, has the time inversion property and satisfies (H1-3), then it necessarily has a generator, when applied to a function $f \in C_0^2(\mathbb{R} \setminus \{0\})$ of the form (2.14) for $\nu \geq 0$. Moreover,

if $\rho \in C^2(\mathbb{R})$, then this restricts the generator of R to

$$\sigma^2 \left[\frac{1}{2} f''(x) + \frac{2\nu + 1}{2x} f'(x) + \frac{q(\text{sgn}(x))}{x^2} (f(-\kappa(\text{sgn}(x))x) - f(x)) \right]. \quad (4.11)$$

We would like to show that this class of processes can be written as a skew product representation. This also provides an alternate proof of the time inversion property for this class of processes.

Corollary 46. *A Feller process R on \mathbb{R} with a generator applied to an $f \in C_0^2(\mathbb{R} \setminus \{0\})$ given by 4.11 with $\sigma = 1$ for a $\nu \geq 0$, $x \neq 0$ has the skew product representation on \mathbb{C} given by $R_t = r_t e^{i\pi N_{H_t}^{\lambda_-, \lambda_+}}$, where r is a time-scaled Bessel process and $N_t^{\lambda_-, \lambda_+}$ is a counting process from the Lamperti-Kiu representation, see Section 1.1.4, with a parameter that depends on the parity of the current position. Since the process has this skew product representation, it also has the time inversion property by Theorem 44.*

Peculiarly, we also note that this is an unorthodox skew product representation since it does not leave a one-dimensional subset of a two-dimensional space and thus, it is a process on \mathbb{R} .

Proof. This follows straightforwardly from Chaumont et al. [2013]. By Volkonski's Theorem, see Volkonski [1958], we can show that R is a Lamperti-Kiu transform of a Feller multiplicative process $K := (K_t)_{t \geq 0}$ with the generator applied to a function $f \in C_0^2(\mathbb{R} \setminus \{0\})$ of the form

$$\mathcal{A}_K f(x) = \frac{x^2}{2} f''(x) + \frac{2\nu + 1}{2} x f'(x) + q(\text{sgn}(x)) (f(-\kappa(\text{sgn}(x))x) - f(x)).$$

The result follows from the generator of the Feller multiplicative process given in [Chaumont et al., 2013, Proposition 7]. \square

4.4 The Time Inversion Property and the Skew Product Representation in Two Dimensions

Now that we have the time inversion property for processes of the skew product form, we take a more detailed look at these processes on \mathbb{R}^2 . In doing so, we also link this skew product expression for processes with the time inversion property with the semigroup densities given in Gallardo and Yor [2005] and Lawi [2008]. As described in Section 4.1, there are two main ways by which examples of processes with the time inversion property have been constructed. Namely, the rotational invariant method of Vuolle-Apiala [2012] and the conditioned semigroup density method of Gallardo and Yor [2005] and Lawi [2008]. Utilising Theorem 44, we compare and contrast the two styles and the sets of processes upon which they apply. We then use the skew product representation to extend the range of these methodologies and construct examples of processes with the time inversion property outside the scope of the existing methods.

This section begins by determining a restricted class of processes in two dimensions through a generalisation of the Hartman-Watson distribution provided in Pitman and Yor [1981] to time-changed Lévy processes. In Section 4.4.2, we conduct a comparison of the two approaches to characterising the time inversion property. Beginning with the approach of Vuolle-Apiala [2012], we show that the rotational invariance property (RI) used as a condition is equivalent to the angular part of the skew product γ being a Lévy process. We then compare the approach of Vuolle-Apiala with the semigroup density approach of Gallardo and Yor [2005] and Lawi [2008]. With the Lévy restriction on the angular part and an additional restriction on the Lévy exponent, we deduce the form of the semigroup density of the processes considered in Vuolle-Apiala [2012] precisely, and in doing so, we show that these processes are a subset of those considered under the restrictions of Gallardo and Yor [2005]. We then look at a few known examples that show this. In the opposite direction, Section 4.4.3 then explores the construction of a few examples of semigroup densities of processes that enjoy the time inversion property and the skew product representation on \mathbb{R}^2 including a process that is outside the scope of Vuolle-Apiala [2012]. In each case we find the semigroup density explicitly. In addition, we provide a methodology by which one can construct a near limitless number of processes with the time inversion property on \mathbb{R}^n of the form considered by Gallardo and Yor [2005] using the semigroup densities of other processes. This is not restricted to the angular part being Lévy or continuous so extends the cases of Vuolle-Apiala [2012]. Finally, in Section 4.4.4, we construct an example of a process with the time inversion property that does not satisfy the assumptions made in either of the approaches.

4.4.1 Determining the Process in Two Dimensions: The Generalised Hartman-Watson Distribution

In order to gain a greater understanding of the class of processes that can be expressed in skew product form in two dimensions, we first concern ourselves with determining the processes that have this property in two dimensions. We do this by first looking at the moments of these processes when the angular part is Lévy.

In restricting to two-dimensions, we have already observed that we can write our skew product representation in the complex plane as $R_t = r_t e^{i\gamma_{H_t}}$, where r is a Bessel process and H_t is the time change (4.1). In order to compute the moment of this process we also make the assumption that the angular part γ is an independent Lévy process, whose Lévy exponent satisfies $2\phi(\lambda) + \nu^2 \geq 0$ and that this expression has no imaginary part for all $\lambda > 0$. These assumptions are assumptions that are satisfied when γ is a Brownian motion. With these assumptions, we can fully determine both the radial and angular parts separately and together through the moments of their joint distribution as a generalisation of the distribution in Hartman and Watson [1974] in the following way.

Proposition 47. *If r is a Bessel process with index $\nu \geq 0$ and γ is an independent Lévy process, whose Lévy exponent is given by a ϕ satisfying $2\phi(\lambda) + \nu^2 \geq 0$ and real for all $\lambda > 0$ that is*

time-changed by $H_t = \int_0^t \frac{1}{r_s^2} ds$, then

$$\mathbb{E}_a \left[r_t^\mu e^{-i\lambda\gamma_{H_t}} \right] = \int_0^\infty I_{\sqrt{2\phi(\lambda)+\nu^2}} \left(\frac{ar}{t} \right) \frac{r^{\nu+\mu+1}}{a^\nu t} e^{-\frac{a^2+r^2}{2t}} dr \quad (4.12)$$

for any $\lambda, \mu > 0$.

Moreover, provided that $\nu + \mu + \sqrt{2\phi(\lambda) + \nu^2} > -2$, this integral can be expressed as

$$\mathbb{E}_a \left[r_t^\mu e^{-i\lambda\gamma_{H_t}} \right] = \frac{e^{-\frac{a^2}{4t}} 2^{\frac{\nu+\mu+1}{2}} t^{\frac{\nu+\mu+1}{2}} \Gamma \left(\frac{\nu+\mu+1+\sqrt{2\phi(\lambda)+\nu^2}+1}{2} \right)}{a^{\nu+1} \Gamma \left(\sqrt{2\phi(\lambda) + \nu^2} + 1 \right)} M_{-\frac{\nu+\mu+1}{2}, \frac{\sqrt{2\phi(\lambda)+\nu^2}}{2}} \left(\frac{a^2}{4t} \right).$$

Here M is the Whittaker function, given in Appendix A.3, and \mathbb{P}_a is the probability measure associated with the skew product representation $r_t e^{i\gamma_{H_t}}$ on \mathbb{C} when the Bessel process is initiated at $a \in \mathbb{R}^+$ and the Lévy process is initiated at $\gamma_0 = 0$.

In addition, the moments of the subject of $R_t = r_t e^{i\gamma_{H_t}}$ on \mathbb{C} , are given by (4.12), when $\mu = \lambda$ provided that $2\phi(\mu) + \nu^2 \geq 0$ and has no imaginary component.

In this proposition we have defined \mathbb{P}_a as the measure of (r_t, γ_t) when r is initiated at $r_0 = a$. We have assumed that γ is initiated at $\gamma_0 = 0$ since the process is Lévy and the case for any other starting points can be easily deduced by the property of stationary increments.

Proof. We first note that a Bessel process with index $\nu \geq 0$ avoids zero almost surely by [Revuz and Yor, 2005, Chapter XI Proposition XI.1.5] so we can continue without giving any thought to possible behaviour at zero and therefore, the result of this proposition can be found through just calculation. If we take the combined moments, then by independence of the Lévy process γ and the law of total probability

$$\mathbb{E}_a \left[r_t^\mu e^{-i\lambda\gamma_{H_t}} \right] = \int_0^\infty r^\mu \mathbb{E}_a \left[e^{-\phi(\lambda)H_t} | r_t = r \right] \mathbb{P}_a(r_t \in dr).$$

Both the semigroup of a Bessel process of index $\nu \geq 0$ and the Laplace transform of the Hartman-Watson distribution in terms of the modified Bessel function I_ν are well known in the case when $2\phi(\lambda) + \nu^2 \geq 0$ and real. If we substitute the expressions given for each of these in Pitman and Yor [1981] into the above and simplify

$$\begin{aligned} \mathbb{E}_a \left[r_t^\mu e^{-i\lambda\gamma_{H_t}} \right] &= \int_0^\infty r^\mu \frac{I_{\sqrt{2\phi(\lambda)+\nu^2}} \left(\frac{ar}{t} \right)}{I_\nu \left(\frac{ar}{t} \right)} I_\nu \left(\frac{ar}{t} \right) \left(\frac{r}{a} \right)^\nu \frac{r}{t} e^{-\frac{a^2+r^2}{2t}} dr \\ &= \int_0^\infty I_{\sqrt{2\phi(\lambda)+\nu^2}} \left(\frac{ar}{t} \right) \frac{r^{\nu+\mu+1}}{a^\nu t} e^{-\frac{a^2+r^2}{2t}} dr \\ &= \frac{e^{-\frac{a^2}{2t}} t^{\nu+\mu+1}}{2a^{2\nu+\mu+2}} \int_0^\infty I_{\sqrt{2\phi(\lambda)+\nu^2}}(\sqrt{w}) w^{\frac{\nu+\mu}{2}} e^{-\frac{wt}{2a^2}} dw. \end{aligned}$$

Here, we made the substitution $w = \frac{a^2 r^2}{t^2}$ inside the integral. Moreover, to simplify this further, we can use the identity for the integrated modified Bessel function given in [Gradshteyn and

Ryzhik, 2007, P709 Formula 6.643], which we recall here

$$\int_0^\infty x^{\tilde{\mu}-\frac{1}{2}} e^{-\alpha x} I_{2\tilde{\nu}}(2\beta\sqrt{x}) dx = \frac{\Gamma(\tilde{\mu} + \tilde{\nu} + \frac{1}{2})}{\Gamma(2\tilde{\nu} + 1)} \beta^{-1} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\tilde{\mu}} M_{-\tilde{\mu}, \tilde{\nu}}\left(\frac{\beta^2}{2\alpha}\right), \quad (4.13)$$

where M is a Whittaker function, as detailed in Appendix A.3. This result applies provided that $\operatorname{Re}(\nu + \mu + \sqrt{2\phi(\lambda) + \nu^2}) > -2$. Therefore, under this restriction we have the result in the statement of the proposition.

The expression for the moments of R given in the statement of the proposition follows by letting $\mu = \lambda$ in the expression of the separated moments. □

This provides us with a way of determining the process represented as a skew product representation through its moments and, provided that the angular part is Lévy up to the time change (4.1), it allows us to relate the semigroup density to an integrated Bessel process.

4.4.2 Comparison of the Two Approaches: Vuolle-Apiala's Approach as a Subset of Gallardo, Yor and Lawi's Approach in Two Dimensions

In this section we would like to compare the two approaches to the time inversion property. On the one hand, Vuolle-Apiala [2012] considers the set of 2-self-similar diffusions on \mathbb{R}^n for $n \geq 2$, which he shows admits the skew product representation (1.9), where Θ is a spherical Brownian motion on S^{n-1} . Thus, in two dimensions this is equivalent to the skew product representation $r_t e^{iB_{Ht}}$, where B is a Brownian motion, on the complex plane \mathbb{C} .

On the other hand, Gallardo and Yor [2005] and Lawi [2008] considered a Markov process, which, in two dimensions, was equivalent to a process R on the complex plane initiated at a point $x \in \mathbb{C}$ with the assumptions (H1), (H2') and (H3) from Section 2.2 applied to the state space $S = \mathbb{C}$.

Under these conditions Gallardo and Yor [2005] and Lawi [2008] were able to show that, up to an h -transform, a process satisfying (H1-3) enjoyed the time inversion property if and only if it had a semigroup density of the form (2.2) for restrictions on the individual functions detailed in Section 2.2 by (2.3), (2.4) and (2.5).

We show that the processes considered by Vuolle-Apiala in two dimensions are a subset of the processes considered by Gallardo, Yor and Lawi in two dimensions. More than this, we explicitly determine the form of the semigroup density of the processes considered by Vuolle-Apiala and show how the general form of the semigroup density associated with the processes considered by Vuolle-Apiala is a restricted form of the semigroup density deduced by Lawi [2008].

The continuous paths property already implies that the rotationally invariant diffusion processes considered by Vuolle-Apiala are a subset of the class of Markov processes considered by Gallardo and Yor and Lawi restricted by (H1-3) since diffusions have continuous semigroup densities. However, we would like to determine the semigroup densities of the class of processes considered by Vuolle-Apiala explicitly. In fact, we know from Theorem 44 that a process with the

skew product representation has the time inversion property for any Feller process representing its angular part. Using this we determine the semigroup density, provided it exists with respect to the Lebesgue measure, of any process with the skew product representation as long as the angular part is a Lévy process with a restriction on the Lévy exponent. This restriction on the Lévy exponent includes the case of Brownian motion implied by Vuolle-Apiala [2012].

Before all else, we determine the exact class of skew product representations that lead to the class of processes that Vuolle-Apiala [2012] showed to enjoy the time inversion property. This is done by showing the equivalence between the rotational invariance property (RI) considered by Vuolle-Apiala [2012] and the Lévy property of the angular part. This Lévy property lends itself to the construction of a semigroup density and therefore the approach of Gallardo and Yor [2005].

Proposition 48. *Let R be a Feller process on \mathbb{C} that admits the skew product representation (4.4), where r is a, possibly time-scaled, Bessel process of index $\nu \geq 0$ and γ is an independent time-homogeneous Feller process. Then R has the rotational invariance property (RI) if and only if the angular part γ is a Lévy process.*

Proof. If we first assume that $R_t = r_t e^{i\gamma_{H_t}}$ is a Feller process with the rotational invariance property then for any Borel f we have that

$$\mathbb{E}_{T^{-1}(x)} [f(T(r_t e^{i\gamma_{H_t}}))] = \mathbb{E}_x [f(r_t e^{i\gamma_{H_t}})]$$

for any T in the group of rotational operators. Firstly, we assert that the rotation T is a rotation of an angle, say θ , in an anticlockwise direction. Taking the polar coordinates of x to be (r_0, γ_0) and noting that the rotation does not affect the radial part of the process given by the Bessel process we have that

$$\mathbb{E}_{r_0, \gamma_0 - \theta} [f(r_t e^{i(\gamma_{H_t} + \theta)})] = \mathbb{E}_{r_0, \gamma_0} [f(r_t e^{i\gamma_{H_t}})],$$

where $\mathbb{P}_{r_0, \gamma_0}$ indicates the measure of a process whose radial part is started at r_0 and angular part at γ_0 . Using the independence of r and γ together with the law of total probability and taking $f(re^{i\psi}) = g(r)e^{i\lambda\psi}$ for any Borel g

$$\begin{aligned} & \int_0^\infty \int_0^\infty g(r) \mathbb{E}_{\gamma_0 - \theta} [e^{i\lambda(\gamma_s + \theta)}] \mathbb{P}_{r_0}(r_t \in dr; H_t \in ds) \\ &= \int_0^\infty \int_0^\infty g(r) \mathbb{E}_{\gamma_0} [f(e^{i\lambda\gamma_s})] \mathbb{P}_{r_0}(r_t \in dr; H_t \in ds) \end{aligned}$$

Letting $\theta = \gamma_0$, we have that for all $\lambda > 0$

$$\mathbb{E}_{\gamma_0} [e^{i\lambda\gamma_t}] = e^{i\lambda\gamma_0} \mathbb{E} [e^{i\lambda\gamma_t}]. \quad (4.14)$$

With this and the Markov, Feller and time homogeneous properties of γ we can now prove the stationary and independent increments of the process. By the tower property of

expectations, for any $\lambda, \mu > 0$

$$\begin{aligned}\mathbb{E} \left[e^{i\lambda(\gamma_t - \gamma_s)} e^{i\mu\gamma_s} \right] &= \mathbb{E} \left[\mathbb{E} \left[e^{i\lambda(\gamma_t - \gamma_s)} | \mathcal{F}_s^\gamma \right] e^{i\mu\gamma_s} \right] \\ &= \mathbb{E} \left[e^{-i\lambda\gamma_s} \mathbb{E} \left[e^{i\lambda\gamma_t} | \mathcal{F}_s^\gamma \right] e^{i\mu\gamma_s} \right],\end{aligned}$$

where \mathcal{F}_s^γ is the σ -algebra generated by $\sigma(\gamma_u : u \leq s)$. Moreover, by the Markov and time homogeneous properties of γ this reduces to

$$\mathbb{E} \left[e^{i\lambda(\gamma_t - \gamma_s)} e^{i\mu\gamma_s} \right] = \mathbb{E} \left[e^{-i\lambda\gamma_s} \mathbb{E}_{\gamma_s} \left[e^{i\lambda\gamma_{t-s}} \right] e^{i\mu\gamma_s} \right]$$

and so by (4.14)

$$\mathbb{E} \left[e^{i\lambda(\gamma_t - \gamma_s)} e^{i\mu\gamma_s} \right] = \mathbb{E} \left[e^{-i\lambda\gamma_s} e^{i\lambda\gamma_s} \mathbb{E} \left[e^{i\lambda\gamma_{t-s}} \right] e^{i\mu\gamma_s} \right] = \mathbb{E} \left[e^{i\lambda\gamma_{t-s}} \right] \mathbb{E} \left[e^{i\mu\gamma_s} \right],$$

which proves the stationary and independent increments properties. The stochastically continuous property follows from the Feller property of the process.

On the other hand if $R_t = r_t e^{i\gamma_{H_t}}$ is a Feller process where γ is Lévy then for any T , an anticlockwise rotation by some $\theta \in [0, 2\pi)$, where x has radial part r_0 and angular part γ_0

$$\begin{aligned}\mathbb{E}_{T^{-1}(x)} \left[f(T(r_t e^{i\gamma_{H_t}})) \right] &= \mathbb{E}_{T^{-1}(r_0, \gamma_0)} \left[f(T(r_t e^{i\gamma_{H_t}})) \right] \\ &= \mathbb{E}_{r_0, \gamma_0 - \theta} \left[f(r_t e^{i\gamma_{H_t} + i\theta}) \right]\end{aligned}$$

by the independence of the radial part under rotation. Now, by the law of total probability and the independence of γ and r ,

$$\mathbb{E}_{T^{-1}(x)} \left[f(T(r_t e^{i\gamma_{H_t}})) \right] = \int_0^\infty \int_0^\infty \mathbb{E}_{\gamma_0 - \theta} \left[f(r e^{i\gamma_s + i\theta}) \right] \mathbb{P}_{r_0}(r_t \in dr; H_t \in ds)$$

and by the Lévy property of γ

$$\begin{aligned}\mathbb{E}_{T^{-1}(x)} \left[f(T(r_t e^{i\gamma_{H_t}})) \right] &= \int_0^\infty \int_0^\infty \mathbb{E}_{\gamma_0} \left[f(r e^{i\gamma_s}) \right] \mathbb{P}_{r_0}(r_t \in dr; H_t \in ds) \\ &= \mathbb{E}_{r_0, \gamma_0} \left[f(r_t e^{i\gamma_{H_t}}) \right],\end{aligned}$$

which shows rotational invariance. □

This shows that the cases considered by Vuolle-Apiala are exactly the cases of the skew product representation (4.3) when γ is a Lévy process with continuous paths. In our aim to determine the semigroup density of this process, we can show that the rotation invariance property alone goes some way to producing the explicit semigroup density and thus, the explicit general form of processes Vuolle-Apiala showed to have the time inversion property.

Proposition 49. *If R is a Markov process on \mathbb{R}^2 with a semigroup density of the form (2.2), then the rotational invariance property (RI) implies that the semigroup density can be restricted*

to

$$p_t(x, y) = \tilde{\Phi} \left(\frac{\|x\| \|y\|}{t}, \psi \right) \frac{\|y\|^\beta}{t^{\frac{\beta}{2}+1}} \exp \left(-\frac{\|x\|^2 + \|y\|^2}{2\sigma^2 t} \right),$$

where ψ is the angle between x and y and $\tilde{\Phi} : \mathbb{R}^+ \times [0, 2\pi) \rightarrow \mathbb{R}^+$.

Proof. By the rotational invariance property (RI), for any rotation $T \in O(n)$ we have

$$p_t(x, y) = p_t(T(x), T(y)),$$

where T is a rotation by some angle $\phi \in [0, 2\pi)$. If we now employ the form of the semigroup density of a process with the time inversion property given in (2.2), this implies that, in polar coordinates for any $r_x, r_y, t > 0$ and $\phi \in [0, 2\pi)$ where ϕ represents the angle of rotation associated with T , the above can be written

$$\begin{aligned} & \Phi \left(\begin{pmatrix} \cos(\theta_x + \phi) \\ \sin(\theta_x + \phi) \end{pmatrix}, \frac{r_x r_y}{t} \begin{pmatrix} \cos(\theta_y + \phi) \\ \sin(\theta_y + \phi) \end{pmatrix} \right) \theta \left(\begin{pmatrix} \cos(\theta_y + \phi) \\ \sin(\theta_y + \phi) \end{pmatrix} \right) \exp \left(-\frac{\rho(T(x)) + \rho(T(y))}{t} \right) \\ &= \Phi \left(\begin{pmatrix} \cos \theta_x \\ \sin \theta_x \end{pmatrix}, \frac{r_x r_y}{t} \begin{pmatrix} \cos \theta_y \\ \sin \theta_y \end{pmatrix} \right) \theta \left(\begin{pmatrix} \cos \theta_y \\ \sin \theta_y \end{pmatrix} \right) \exp \left(-\frac{\rho(x) + \rho(y)}{t} \right), \end{aligned}$$

where we have cancelled the r_y and t terms appearing on either side. If we now define the function $\hat{\Phi}$ as follows

$$\begin{aligned} & \hat{\Phi} \left(\begin{pmatrix} \cos(\theta_x) \\ \sin(\theta_x) \end{pmatrix}, \frac{r_x r_y}{t} \begin{pmatrix} \cos(\theta_y) \\ \sin(\theta_y) \end{pmatrix} \right) \\ &= \Phi \left(\begin{pmatrix} \cos(\theta_x) \\ \sin(\theta_x) \end{pmatrix}, \frac{r_x r_y}{t} \begin{pmatrix} \cos(\theta_y) \\ \sin(\theta_y) \end{pmatrix} \right) \theta \left(\begin{pmatrix} \cos(\theta_y) \\ \sin(\theta_y) \end{pmatrix} \right), \end{aligned}$$

to make the notation slightly more legible, then for all $r_x = 1$ and $r_y = t$ we obtain

$$\begin{aligned} & \hat{\Phi} \left(\begin{pmatrix} \cos(\theta_x + \phi) \\ \sin(\theta_x + \phi) \end{pmatrix}, \begin{pmatrix} \cos(\theta_y + \phi) \\ \sin(\theta_y + \phi) \end{pmatrix} \right) e^{-\frac{\rho(\hat{x})}{t}} e^{-t\rho(\hat{y})} \\ &= \hat{\Phi} \left(\begin{pmatrix} \cos \theta_x \\ \sin \theta_x \end{pmatrix}, \begin{pmatrix} \cos \theta_y \\ \sin \theta_y \end{pmatrix} \right) e^{-\frac{\rho(\hat{x})}{t}} e^{-t\rho(\hat{y})}. \end{aligned}$$

Moreover, as this applies for any value of $t > 0$, we can equate the parts that have the same dependence on t and therefore, since $\rho(\hat{y}) = \rho(T\hat{y})$ for any rotation T ,

$$\rho(y) = \|y\|^2 \rho(\hat{y}) = \frac{\|y\|^2}{2\sigma^2}$$

and also, taking the equality of parts with no dependence on t for any θ_y, θ_x and $\phi \in S^{n-1}$,

gives

$$\hat{\Phi}(x, y) = \tilde{\Phi}(\|x\| \|y\|, \theta_y - \theta_x).$$

□

Immediately, it becomes apparent that $\tilde{\Phi}(\cdot, \cdot)$ is all that is left to resolve to completely determine the semigroup density. In order to determine this function we require that the angular component is Lévy and has a Lévy exponent ϕ that satisfies $2\phi(\lambda) + \nu^2 \geq 0$ and real for all $\lambda > 0$, where $\nu \geq 0$ is the index of the Bessel process representing the radial part. We note that this includes the case considered by Vuolle-Apiala [2012]. Once we have these, we show that we can determine ϕ through a Fourier transform.

Theorem 50. *Let R be a process satisfying (H1-3) with the time inversion property that can be written as a skew representation, as in Definition 41, with the additional condition that the angular part γ is Lévy and has a Lévy exponent ϕ that satisfies $2\phi(\lambda) + \nu^2 \geq 0$ and real for all $\lambda > 0$ and the radial part is a Bessel process in standard scale. Then $\tilde{\Phi}$ in the semigroup density of this process can be determined through the Fourier transform in terms of the modified Bessel function I_ν as*

$$\int_0^{2\pi} e^{-i\lambda\psi} \tilde{\Phi}(z, \psi) d\psi = I_{\sqrt{2\phi(\lambda) + \nu^2}}(z) z^{-\nu},$$

where ϕ is the Lévy exponent of the process γ and ν is the index of the Bessel process underlying the skew product.

Proof. We shall determine the function $\tilde{\Phi}$ in the semigroup density of the process by using the restriction on the semigroup in Proposition 49 combined with the expression of the Laplace transform of the Hartman-Watson distribution given in Pitman and Yor [1981]. In this way, we can isolate the angular part of the function $\tilde{\Phi}$ using the joint distribution of the angular part and the fact that the radial part is a Bessel process. The radial part of the function $\tilde{\Phi}$ then follows from the Bessel properties of the radial part of the process.

To begin with, we take this opportunity to note the difference between the semigroup density of γ_{H_t} and the part of this semigroup density in the angular part of R that is modulo 2π . That is,

$$\mathbb{P}_{r_x, \gamma_0}(\gamma_{H_t} \in d\phi; r_t \in dr_y) = \Phi^\gamma\left(\frac{r_x r_y}{t}, \phi\right) \frac{r_y^{\beta+1}}{t^{\frac{\beta}{2}+1}} e^{-\frac{r_x^2 + r_y^2}{2t}} d\phi dr_y,$$

where $\tilde{\Phi}$ is given by

$$\tilde{\Phi}\left(\frac{r_x r_y}{t}, \phi\right) = \sum_{n=-\infty}^{\infty} \Phi^\gamma\left(\frac{r_x r_y}{t}, 2n\pi + \phi\right). \quad (4.15)$$

In this way, we can see that the angular part of the semigroup density of the skew

product representation has an equivalence with the density of the process that describes the angular part, but its state space is modulo 2π .

We now determine $\tilde{\Phi}$ in the density of the skew product representation explicitly using the known properties of the Hartman-Watson distribution. Utilising the skew product representation of the process on the complex plane

$$\mathbb{E}_{r_0} [f(r_t e^{i\gamma_{H_t}})] = \int_0^\infty \int_0^\infty f(re^{i\psi}) \mathbb{P}_{r_0}(\gamma_{H_t} \in d\psi; r_t = r) dr$$

and using our expression for the rotationally invariant semigroup density of Proposition 49, where we note that the integrating over the entire real line is equivalent to integrating over a countable sum of $[0, 2\pi)$ intervals

$$\mathbb{E}_{r_0} [f(r_t e^{i\gamma_{H_t}})] = \int_0^\infty \int_{-\infty}^\infty f(re^{i\psi}) \Phi^\gamma\left(\frac{rxry}{t}, \psi\right) \frac{r_y^{\beta+1}}{t^{\frac{\beta}{2}+1}} \exp\left(-\frac{r_x^2 + r_y^2}{2t}\right) d\theta_y dr_y \quad (4.16)$$

$$= \int_0^\infty \int_0^{2\pi} f(re^{i\psi}) \tilde{\Phi}\left(\frac{rxry}{t}, \psi\right) \frac{r_y^{\beta+1}}{t^{\frac{\beta}{2}+1}} \exp\left(-\frac{r_x^2 + r_y^2}{2t}\right) d\theta_y dr_y, \quad (4.17)$$

where we have used the expression for $\tilde{\Phi}$ given in (4.15) above.

Provided that γ is Lévy and satisfies $2\phi(\lambda) + \nu^2 \geq 0$ and real, we can find the Fourier transform of the conditioned and time-changed process. Since γ is Lévy, we also assume that it starts at $\gamma_0 = 0$ because it is then easy to extrapolate from there by the stationary increments of the process. By the independence of γ with respect to r and H_t we can express γ in terms of its Lévy exponent

$$\mathbb{E}_{r_0} [e^{-i\lambda\gamma_{H_t}} | r_t = r] = \mathbb{E}_{r_0, \gamma_0=0} [e^{-\phi(\lambda)H_t} | r_t = r],$$

where ϕ is the Lévy exponent of γ . Using a result from Pitman and Yor [1981] we used in Proposition 47, we can now express the Fourier transform in terms of the modified Bessel functions

$$\begin{aligned} \frac{I_{\sqrt{2\phi(\lambda)+\nu^2}}\left(\frac{r_0 r}{t}\right)}{I_\nu\left(\frac{r_0 r}{t}\right)} &= \mathbb{E}_{r_0} [e^{-i\lambda\gamma_{H_t}} | r_t = r] \\ &= \int e^{-i\lambda\psi} \mathbb{P}_{r_0}(\gamma_{H_t} \in d\psi | r_t = r). \end{aligned}$$

By the conditional distributions in Definition 42, we can write this as

$$\frac{I_{\sqrt{2\phi(\lambda)+\nu^2}}\left(\frac{r_0 r}{t}\right)}{I_\nu\left(\frac{r_0 r}{t}\right)} = \int_0^{2\pi} e^{-i\lambda\psi} \frac{\mathbb{P}_{r_0}(\gamma_{H_t} \in d\psi; r_t \in dr)}{\mathbb{P}_{r_0}(r_t \in dr)}$$

and by (4.16) we know that the numerator can be written

$$\frac{I_{\sqrt{2\phi(\lambda)+\nu^2}}\left(\frac{r_0 r}{t}\right)}{I_\nu\left(\frac{r_0 r}{t}\right)} = \int_0^\infty e^{-i\lambda\psi} \frac{\tilde{\Phi}\left(\frac{r_x r_y}{t}, \psi\right) \frac{r_y^{\beta+1}}{t^{\frac{\beta}{2}+1}} \exp\left(-\frac{r_x^2+r_y^2}{2t}\right) d\psi}{\mathbb{P}_{r_0}(r_t \in dr)}.$$

Furthermore, the semigroup density of the Bessel process is also known (see [Revuz and Yor, 2005, Chapter XI Corollary XI.1.4])

$$\frac{I_{\sqrt{2\phi(\lambda)+\nu^2}}\left(\frac{r_0 r}{t}\right)}{I_\nu\left(\frac{r_0 r}{t}\right)} = \int_0^{2\pi} e^{-i\lambda\psi} \frac{\tilde{\Phi}\left(\frac{r_x r_y}{t}, \psi\right) \frac{r_y^{\beta+1}}{t^{\frac{\beta}{2}+1}} \exp\left(-\frac{r_x^2+r_y^2}{2t}\right) d\psi}{I_\nu\left(\frac{r_0 r}{t}\right) \frac{r_0^{\nu+2}}{r_0^\nu t} \exp\left(-\frac{r_x^2+r_y^2}{2t}\right)}.$$

Cancelling and simplifying this expression yields

$$I_{\sqrt{2\phi(\lambda)+\nu^2}}\left(\frac{r_0 r}{t}\right) = \int_0^{2\pi} e^{-i\lambda\psi} \tilde{\Phi}\left(\frac{r_x r_y}{t}, \psi\right) \frac{r_y^{\beta-\nu} r_0^\nu}{t^{\frac{\beta}{2}}} d\psi.$$

If we compare each side we notice that the left hand side only has dependence on $\frac{r_0 r}{t}$ not r_y or t separately. This suggests that $\beta = 2\nu$ on the right hand side and therefore if we let $z = \frac{r_x r_y}{t}$

$$I_{\sqrt{2\phi(\lambda)+\nu^2}}(z) z^{-\nu} = \int_0^{2\pi} e^{-i\lambda\psi} \tilde{\Phi}(z, \psi) d\psi.$$

□

This result now allows us to entirely characterise all processes, which Vuolle-Apiala showed to have the time inversion property, through their semigroup density.

We can check this result via the conservative property of the process when $\lambda = 0$.

Example 51. When $\lambda = 0$ in the Theorem 50, we expect the result

$$\int_0^{2\pi} \tilde{\Phi}(z, \psi) d\psi = I_\nu(z) z^{-\nu}.$$

By using the conservative property of the process and the semigroup density involving $\hat{\Phi}$ derived in polar coordinates above we have

$$1 = \int_0^\infty \int_0^{2\pi} \tilde{\Phi}\left(\frac{r r_0}{t}, \psi\right) \frac{r^{2\nu+1}}{t^{\nu+1}} e^{-\frac{r_0^2+r^2}{2t}} d\psi dr.$$

If we now take out the terms not involved in the integral and make the substitution $\frac{r r_0}{t} = 2\sqrt{u}$

$$e^{\frac{r_0^2}{2t}} = \int_0^\infty \int_0^{2\pi} \tilde{\Phi}(2\sqrt{u}, \psi) d\psi \frac{2^{2\nu+1} t^{\nu+1}}{r_0^{2\nu+2}} u^\nu e^{-\frac{2tu}{r_0^2}} du$$

and letting $\mu = \frac{r_0^2}{2t}$

$$\frac{e^{\frac{1}{\mu}}}{\mu^{\nu+1}} = \int_0^\infty \int_0^{2\pi} \tilde{\Phi}(2\sqrt{u}, \psi) d\psi 2^\nu u^\nu e^{-\mu u} du$$

and therefore, by the Laplace transform of a Bessel function, given in [Gradshteyn and Ryzhik, 2007, p. 709 Formula 6.643.2], we have

$$\int_0^{2\pi} \tilde{\Phi}(z, \psi) d\psi = I_\nu(z) 2z^{-\nu}.$$

This implies that the theorem holds in the case when $\lambda = 0$.

We can now look at example semigroup densities of specific processes with the time inversion property in two dimensions and we can determine the function $\tilde{\Phi}$ explicitly.

Example 52 (Two-Dimensional Brownian Motion). *We know that two-dimensional Brownian motion is a Feller process satisfying (H1-4) that also enjoys the time inversion property. Furthermore, by Le Gall [1992], it is also known that it has the skew product representation $r_t e^{iB_{H_t}}$, where r is a two-dimensional Bessel process as above and B is an independent one-dimensional Brownian motion. Thus, its Lévy exponent is given by $\phi(\lambda) = \frac{\lambda^2}{2}$. Importantly, we also know that the semigroup density of this process can be given in the form in Gallardo and Yor [2005] as*

$$p_t^B(x, y) = \frac{1}{\sqrt{(2\pi)^2 t}} \exp\left(\frac{x \cdot y}{t}\right) \exp\left(-\frac{\|x\|^2 + \|y\|^2}{2t}\right)$$

and so, in polar coordinates

$$\begin{aligned} \Phi^B\left(x, \frac{y}{t}\right) &= \frac{1}{2\pi} \exp\left(\frac{x \cdot y}{t}\right) \\ &= \frac{1}{2\pi} \exp\left(\frac{x_1 \cdot y_1 + x_2 \cdot y_2}{t}\right) \\ &= \frac{1}{2\pi} \exp\left(\frac{rr_0}{t} \cos(\psi - \gamma_0)\right). \end{aligned}$$

for $y = (r, \psi)$, $x = (r_0, \gamma_0)$. Thus, we can denote the function $\tilde{\Phi}^B$ in two-dimensional Brownian motion as

$$\hat{\Phi}^B(z, \psi) = \frac{1}{2\pi} \exp(z \cos(\psi)).$$

If we now combine this with the Lévy exponent of Brownian motion, $\phi(\lambda) = \frac{\lambda^2}{2}$ and the index

of the Bessel process, $\nu = 0$,

$$I_{\sqrt{2\phi(\lambda)+\nu^2}}(z) z^{-\nu} = \int_0^{2\pi} e^{-i\lambda\psi} \tilde{\Phi}^B(z, \psi) d\psi$$

$$I_\lambda(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\lambda\psi} \exp(z \cos(\psi)) d\psi,$$

which corroborates with the commonly known expression for the modified Bessel function as an integrand

$$I_\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\psi} \exp(z \cos(\psi)) d\psi.$$

4.4.3 Examples and a Method for Constructing Processes with the Time Inversion Property

The overall aim of this section is to provide a methodology by which one can construct semigroup densities of processes with the time inversion property from a Bessel process and a Feller process with a known semigroup density.

We first consider a few examples of processes in two dimensions that can be expressed by a skew product representation. Importantly, these processes have the time inversion property and therefore, provided that they satisfy (H1-3), we can express them as a semigroup density that coincides with the semigroup density in Gallardo and Yor [2005]. In each of the following examples we show that this is the case and discuss a few properties of the resulting processes.

The Skew Product Representation when γ is Deterministic Drift

We now take a specific example where the angular part of our skew product process is a deterministic drift, that is, $\gamma_{H_t} = dH_t$ for $d > 0$. This draws many similarities with the Hartman and Watson [1974] distribution, which has been studied greatly and, on account of this, we aim to avoid the inverse Fourier transform in the computation of the semigroup density. We also note that this restriction conveys continuous paths to the process and so we should have the absolute continuity property of our semigroup density with respect to the Lebesgue measure (H1).

Once again we begin with the Laplace transform in Pitman and Yor [1981] mentioned in Section 4.1 and we assume that the time change H_t is initiated at $H_0 = 0$ and $\nu \geq 0$ is the index of the Bessel process r . The conditional density of the angular part is given by

$$\frac{I_{\sqrt{2\lambda d + \nu^2}}\left(\frac{rr_0}{t}\right)}{I_\nu\left(\frac{rr_0}{t}\right)} = \mathbb{E}_{r_0} \left[e^{-\lambda(dH_t)} | r_t = r \right] \quad (4.18)$$

$$= \int_0^\infty e^{-\lambda\psi} \mathbb{P}_{r_0}(dH_t \in d\psi | r_t = r). \quad (4.19)$$

However, using the integral identity in Yor [1980], we can express the Bessel function in

the numerator in the following way

$$I_{\sqrt{2\lambda d + \nu^2}}\left(\frac{rr_0}{t}\right) = \int_0^\infty e^{-d\lambda s - \frac{\nu^2}{2}s} \theta\left(\frac{r_0 r}{t}, s\right) ds, \quad (4.20)$$

where the function θ is the function that is referred to in Matsumoto and Yor [2005a], which we recall is given by

$$\theta(r, s) = \frac{r}{\sqrt{2\pi^3 s}} e^{\frac{\pi^2}{2s}} \int_0^\infty e^{-\frac{w^2}{2s}} e^{-r \cosh(w)} \sinh(w) \sin\left(\frac{\pi w}{s}\right) dw.$$

If we make the substitution $ds = \psi$ in the integral identity for the modified Bessel function (4.20), then we obtain

$$I_{\sqrt{2\lambda d + \nu^2}}\left(\frac{rr_0}{t}\right) = \int_0^\infty e^{-\lambda\psi - \frac{\nu^2}{2d}\psi} \theta\left(\frac{r_0 r}{t}, \frac{\psi}{d}\right) \frac{1}{d} d\psi$$

and so returning to equation (4.18)

$$\mathbb{E}_{r_0} \left[e^{-\lambda(dH_t)} | r_t = r \right] = \frac{1}{I_\nu\left(\frac{r_0 r}{t}\right)} \int_0^\infty e^{-\lambda\psi - \frac{\nu^2}{2\xi}\psi} \theta\left(\frac{r_0 r}{t}, \frac{\psi}{d}\right) \frac{1}{d} d\psi.$$

Now that we have an expression for the semigroup in terms of a Laplace transform, we can write the joint probability distribution of the radial and angular parts

$$\mathbb{P}_{r_0}(dH_t \in d\psi | r_t \in dr) \mathbb{P}_{r_0}(r_t \in dr) = e^{-\frac{\nu^2}{2d}\psi} \theta\left(\frac{r_0 r}{t}, \frac{\psi}{d}\right) \frac{1}{d} d\psi \frac{r^{\nu+1}}{r_0^\nu t} e^{-\frac{r_0^2 + r^2}{2t}} dr,$$

or equivalently,

$$\mathbb{P}_{r_0}(dH_t \in d\psi; r_t \in dr) = e^{-\frac{\nu^2}{2d}\psi} \theta\left(\frac{r_0 r}{t}, \frac{\psi}{d}\right) \frac{1}{d} d\psi \frac{r^{\nu+1}}{r_0^\nu t} e^{-\frac{r_0^2 + r^2}{2t}} dr. \quad (4.21)$$

However, we would like to find the distribution of R expressed in its Cartesian coordinates in order to find whether it corroborates with the results of Gallardo and Yor [2005] and Lawi [2008]. Thus, we express the distribution in the following way. For any $r_0 > 0$ and any Borel f with compact support

$$\begin{aligned} & \mathbb{E}_{r_0} [f(r_t \cos(dH_t), r_t \sin(dH_t))] \\ &= \int_0^\infty \int_0^\infty f(r \cos \psi, r \sin \psi) \mathbb{P}_{r_0}(dH_t \in d\psi; r_t \in dr) \\ &= \int_0^\infty \int_0^\infty f(r \cos \psi, r \sin \psi) e^{-\frac{\nu^2}{2d}\psi} \theta\left(\frac{r_0 r}{t}, \frac{\psi}{d}\right) \frac{1}{d} d\psi \frac{r^{\nu+1}}{r_0^\nu t} e^{-\frac{r_0^2 + r^2}{2t}} dr, \end{aligned}$$

where we have substituted in our expression for the semigroup density in polar coordinates

(4.21). Since the interval can be written as a union of intervals $(0, \infty) = \cup_{n=0}^{\infty} (2n\pi, 2(n+1)\pi)$,

$$\begin{aligned}
& \mathbb{E}_{r_0} [f(r_t \cos(dH_t), r_t \sin(dH_t))] \\
&= \sum_{n=0}^{\infty} \int_0^{2\pi} \int_0^{\infty} f(r \cos \psi, r \sin \psi) e^{-\frac{\nu^2}{2d}(\psi+2n\pi)} \theta\left(\frac{r_0 r}{t}, \frac{(\psi+2n\pi)}{d}\right) \frac{1}{d} d\psi \frac{r^{\nu+1}}{r_0^{\nu} t} e^{-\frac{r_0^2+r^2}{2t}} dr \\
&= \int_0^{2\pi} \int_0^{\infty} f(r \cos \psi, r \sin \psi) \sum_{n=0}^{\infty} e^{-\frac{\nu^2 n\pi}{d}} e^{-\frac{\nu^2}{2d}\psi} \theta\left(\frac{r_0 r}{t}, \frac{(\psi+2n\pi)}{d}\right) \frac{1}{d} \frac{r^{\nu}}{r_0^{\nu} t} e^{-\frac{r_0^2+r^2}{2t}} r dr d\psi \\
&= \int \int f(y_1, y_2) \sum_{n=0}^{\infty} e^{-\frac{\nu^2 n\pi}{d}} e^{-\frac{\nu^2}{2d}\psi} \theta\left(\frac{\|x\|\|y\|}{t}, \frac{(\psi+2n\pi)}{d}\right) \frac{1}{d} \frac{\|y\|^{\nu}}{\|x\|^{\nu} t} e^{-\frac{\|x\|^2+\|y\|^2}{2t}} dy_1 dy_2.
\end{aligned}$$

Here, we have exchanged the sum and the integral because all terms are positive and returned to Cartesian coordinates.

We have split the integral into a sum in order to move from polar to Euclidean coordinates and $\psi = \tan^{-1}\left(\frac{y_2}{y_1}\right)$. This implies that the semigroup density of R can be written

$$\begin{aligned}
p_t(x, y) &= \sum_{n=0}^{\infty} e^{-\frac{\nu^2 n\pi}{d}} e^{-\frac{\nu^2}{2d}\psi} \theta\left(\frac{\|x\|\|y\|}{t}, \frac{(\psi+2n\pi)}{d}\right) \frac{1}{d} \frac{\|y\|^{\nu}}{\|x\|^{\nu} t} e^{-\frac{\|x\|^2+\|y\|^2}{2t}} \\
&= \left[\sum_{n=0}^{\infty} e^{-\frac{\nu^2 n\pi}{d}} e^{-\frac{\nu^2}{2d}\psi} \theta\left(\frac{\|x\|\|y\|}{t}, \frac{(\psi+2n\pi)}{d}\right) \frac{1}{d} \frac{t^{\nu}}{\|x\|^{\nu} \|y\|^{\nu}} \right] \frac{\|y\|^{2\nu}}{t^{\nu+1}} e^{-\frac{\|x\|^2+\|y\|^2}{2t}}.
\end{aligned}$$

This is equivalent to the Gallardo and Yor [2005] representation of a semigroup density with the time inversion property with

$$\begin{aligned}
\Phi(x, y) &= \sum_{n=0}^{\infty} e^{-\frac{\nu^2 n\pi}{d}} e^{-\frac{\nu^2}{2d} \tan^{-1}\left(\frac{y_2}{y_1}\right)} \theta\left(\|x\|\|y\|, \frac{(\tan^{-1}\left(\frac{y_2}{y_1}\right) + 2n\pi)}{d}\right) \frac{1}{d} \frac{1}{\|x\|^{\nu} \|y\|^{\nu}} \\
\tilde{\theta}(y) &= \|y\|^{2\nu} \\
\rho(y) &= \frac{\|y\|^2}{2}.
\end{aligned}$$

where Φ , θ and ρ are the functions in (2.2).

The Skew Product Representation when γ is a Cauchy Process

Having considered the simple process where the angular part is deterministic, we would like to include a process with jumps as an example of a process with the time inversion property. This provides an example outside the scope of Vuolle-Apiala [2012].

We explore the case where the self-similar process in two dimensions R is given by $R_t = (r_t \cos C_{H_t}, r_t \sin C_{H_t})$ where $C := (C_t)_{t \geq 0}$ is a symmetric Cauchy process. The Cauchy process is a 1-stable process that has a Cauchy distribution over a one-step time interval. That

is to say, its semigroup is given in [Walck, 2007, Chapter 7] as

$$\mathbb{P}_{C_0}(C_t \in dy) = \frac{1}{\pi} \frac{t}{(y - C_0)^2 + t^2} dy. \quad (4.22)$$

We can find the semigroup density of $R_t = (r_t \cos C_{H_t}, r_t \sin C_{H_t})$ in polar coordinates in this case by using the law of total probability and the independence between C and r . By this token, we can just use the Lévy property and our earlier expressions for the two semigroup densities,

$$\begin{aligned} \mathbb{P}_{r_0, C_0}(C_{H_t} \in d\psi; r_t \in dr) &= \int_0^\infty \mathbb{P}(C_s \in d(\psi - C_0)) \mathbb{P}_{r_0}(H_t \in ds; r_t \in dr) \\ &= \int_0^\infty \frac{1}{\pi} \frac{s}{(\psi - C_0)^2 + s^2} e^{-\frac{\nu^2}{2}s\theta\left(\frac{r_0 r}{t}, s\right)} \frac{r^{\nu+1}}{r_0^\nu t} e^{-\frac{r_0^2 + r^2}{2t}} ds d\psi dr. \end{aligned}$$

Here, we have used the expression for the distribution in polar coordinates given in (4.21) with $d = 1$ for the drift coefficient and the semigroup density of the Cauchy distribution (4.22).

Equivalently, we can also find the semigroup density in Cartesian coordinates in the same way as when γ was deterministic. By following the same method as the previous example, for a Borel f on \mathbb{R}^2

$$\begin{aligned} \mathbb{E}_{r_0}[f(r_t \cos C_{H_t}, r_t \sin C_{H_t})] &= \int_0^\infty \int_{-\infty}^\infty f(r \cos \psi, r \sin \psi) \mathbb{P}_{r_0, C_0}(C_{H_t} \in d\psi; r_t \in dr) \\ &= \int_0^\infty \int_{-\infty}^\infty f(r \cos \psi, r \sin \psi) \int_0^\infty \frac{1}{\pi} \frac{s}{(\psi - C_0)^2 + s^2} e^{-\frac{\nu^2}{2}s\theta\left(\frac{r_0 r}{t}, s\right)} \frac{r^{\nu+1}}{r_0^\nu t} e^{-\frac{r_0^2 + r^2}{2t}} ds d\psi dr \\ &= \sum_{n=-\infty}^\infty \int_0^{2\pi} \int_0^\infty f(r \cos \psi, r \sin \psi) \int_0^\infty \frac{1}{\pi} \frac{s}{(\psi + 2n\pi - C_0)^2 + s^2} e^{-\frac{\nu^2}{2}s\theta\left(\frac{r_0 r}{t}, s\right)} \frac{r^{\nu+1}}{r_0^\nu t} e^{-\frac{r_0^2 + r^2}{2t}} ds d\psi dr \\ &= \int_0^{2\pi} \int_0^\infty f(r \cos \psi, r \sin \psi) \sum_{n=-\infty}^\infty \int_0^\infty \frac{1}{\pi} \frac{s}{(\psi + 2n\pi - C_0)^2 + s^2} e^{-\frac{\nu^2}{2}s\theta\left(\frac{r_0 r}{t}, s\right)} \frac{r^\nu}{r_0^\nu t} e^{-\frac{r_0^2 + r^2}{2t}} ds r d\psi dr \\ &= \int \int f(y_1, y_2) \sum_{n=-\infty}^\infty \int_0^\infty \frac{1}{\pi} \frac{s}{(\psi + 2n\pi - C_0)^2 + s^2} e^{-\frac{\nu^2}{2}s\theta\left(\frac{\|x\|\|y\|}{t}, s\right)} \frac{\|y\|^\nu}{\|x\|^\nu t} e^{-\frac{\|x\|^2 + \|y\|^2}{2t}} ds dy_1 dy_2. \end{aligned}$$

Thus, since this is true for all Borel f we have the semigroup density of the process in Cartesian coordinates

$$\begin{aligned} \mathbb{P}_{x_1, x_2}(R_t \in d(y_1, y_2)) &= \sum_{n=-\infty}^\infty \int_0^\infty \frac{1}{\pi} \frac{s}{(\psi + 2n\pi - C_0)^2 + s^2} e^{-\frac{\nu^2}{2}s\theta\left(\frac{\|x\|\|y\|}{t}, s\right)} ds \frac{\|y\|^\nu}{\|x\|^\nu t} e^{-\frac{\|x\|^2 + \|y\|^2}{2t}} dy_1 dy_2, \end{aligned}$$

where $\psi = \tan^{-1}\left(\frac{y_2}{y_1}\right)$ and $C_0 = \tan^{-1}\left(\frac{x_2}{x_1}\right)$.

This is equivalent to a Gallardo and Yor [2005] representation of a semigroup density

with the time inversion property with

$$\begin{aligned}\Phi(x, y) &= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{s}{(\psi + 2n\pi - C_0)^2 + s^2} e^{-\frac{\nu^2}{2}s} \theta(\|x\|\|y\|, s) \, ds \frac{1}{\|x\|^\nu \|y\|^\nu} \\ \tilde{\theta}(y) &= \|y\|^{2\nu} \\ \rho(y) &= \frac{\|y\|^2}{2}.\end{aligned}$$

Remark 53 (A Methodology for Constructing New Processes with the Time Inversion Property in n -Dimensions). *From the construction of the skew product representation with a one-dimensional Cauchy process for the angular part we take this opportunity to remark that this methodology for producing semigroup densities of processes with the time inversion property is easily extended to n -dimensions. Provided that each component $\gamma_t^{(i)}$ of the angular part $\underline{\gamma}$ is absolutely continuous with respect to the Lebesgue measure then by using the spherical coordinates notation the full semigroup density of the process R can be found as an infinite sum over the individual semigroup densities. This provides us with a technique for constructing limitless examples of processes with the time inversion property through their explicit semigroup density. In each case it can also be seen that, provided the underlying Bessel process of the skew product representation is in natural scale, the functions θ and ρ in (2.2) remain the same as:*

$$\begin{aligned}\tilde{\theta}(y) &= \|y\|^{2\nu+2-n} \\ \rho(y) &= \frac{\|y\|^2}{2}\end{aligned}$$

with only the function Φ changing.

4.4.4 An Example of a Process Enjoying the Time Inversion Property that does not have an Absolutely Continuous Semigroup Density with Respect to the Lebesgue Measure

All the previous known examples of processes enjoying the time inversion property also have a semigroup density that is absolutely continuous with respect to the Lebesgue measure. We would now like to explore an example that satisfies the time inversion property but does not satisfy this hypothesis.

As our proposed example, we take the process with the skew product representation

$$R_t = r_t e^{\frac{i\pi}{2} N_t^{(\lambda)}}$$

on \mathbb{C} , where r is a Bessel process with index $\nu \geq 0$; $N_t^{(\lambda)}$ is a Poisson process with parameter λ and H_t is the time-change given by (4.1) with r restricted to a Bessel process in the natural scale.

As a process for which the skew product representation exists, this process has the time inversion property by the Feller property of the Poisson process and Theorem 44. Thus, we just

need to show that this process is singular with respect to the Lebesgue measure, that is, we would like to find a set A such that the Lebesgue measure of A is zero, but under the probability measure of this process, given by \mathbb{P}_x , the measure of A is strictly greater than zero.

For our set we choose the two perpendicular axes in two dimensions given by the lines $L_1 := \{x_2 = 0\}$ and $L_2 := \{x_1 = 0\}$. As one-dimensional lines in a two-dimensional space the Lebesgue measure of each of these lines can be seen to be zero

$$\text{Lebesgue}(L_1) = \int \int \mathbf{1}_{[(x_1, x_2) \in L_1]} dx_1 dx_2 = \int_{x_1=-\infty}^{\infty} \text{Lebesgue}(\{0\}) dx_1 = 0$$

and similarly for the second line. The union therefore has measure zero by the properties of measures.

$$\text{Lebesgue}(L_1 \cup L_2) \leq \text{Lebesgue}(L_1) + \text{Lebesgue}(L_2) = 0.$$

However, the probability measure of our process on this set is given by

$$\mathbb{P}_x \left(r_t e^{\frac{i\pi}{2} N_{H_t}^{(\lambda)}} \in L_1 \cup L_2 \right).$$

In polar coordinates, the lines are given by $L_1 = \{r \in [0, \infty) \times \theta \in \{0, \pi\}\}$ and $L_2 = \{r \in [0, \infty) \times \theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}\}$. Thus, by taking the polar coordinates of the process the measure of the set is equivalent to

$$\mathbb{P}_x \left(r_t \in [0, \infty); \frac{\pi}{2} N_{H_t}^{(\lambda)} \mod 2\pi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\} \right).$$

Since the Poisson process takes only integer values, the probability of this set is one and we are done.

Appendix A

Special Functions

In an extensive amount of the computations made in this thesis we make use of a variety of special functions and their associated properties. We consider several of the more important and more commonly used properties of these functions here.

A.1 The Modified Bessel Functions - I_ν and K_ν

The modified Bessel function, which is given by $I_\nu(z)$, is an important component of the semi-group density of the Bessel process of index ν for $\nu > -1$ and it results as the increasing solution to the modified Bessel differential equation

$$x^2 \frac{dy}{dx} + x \frac{dy}{dx} - (x^2 + \nu^2)y = 0. \quad (\text{A.1})$$

This is similar to the original Bessel ordinary differential equation solved by $J_\nu(x)$, however, the change in the coefficient of y means that the solution involving J_ν takes an imaginary form and thus, we consider the function $I_\nu(x) = e^{-\frac{i\pi\nu}{2}} J_\nu(ix)$, which provides us with a real solution to the ordinary differential equation (A.1), see [Abramowitz and Stegun, 1972, p. 374 Chapter 9 Equation 9.6.1]. Furthermore, this implies that we can denote the increasing modified Bessel function as the convergent infinite sum

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \nu + 1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

by simply using the definition of $J_\nu(x)$. The decreasing solution to this ODE is known as the modified Bessel function of second kind $K_\nu(x)$. This can be given in terms of I_ν by

$$K_\nu(z) = \frac{\pi}{2 \sin(\nu\pi)} [I_{-\nu}(z) + I_\nu(z)],$$

see [Abramowitz and Stegun, 1972, Chapter 9 Equation 9.6.2].

More information on the modified Bessel function and its properties is available in Watson [1966] and Andrews et al. [1999].

A.2 The Gaussian Hypergeometric Functions - ${}_2F_1$

The Gaussian hypergeometric function is a solution to the differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (c - (a+b+1)x)\frac{dy}{dx} - aby = 0$$

and can be expressed as the sum

$${}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}$$

where $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$ is the Pochhammer symbol, see [Abramowitz and Stegun, 1972, p. 256].

The generality of the differential equation means that many other special functions can be expressed through the hypergeometric function through various substitutions, most notably, the Legendre function.

A.3 The Whittaker Function

Another passably used special function in this thesis is the Whittaker function. In the same manner as many of the special functions employed in this thesis it is also a result of an ordinary differential equation known as the Whittaker differential equation, detailed in [Abramowitz and Stegun, 1972, p. 505 Section 13.1 Equation 13.1.31],

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(\frac{k}{x} + \frac{\frac{1}{4} - m}{x^2} \right) y = 0. \quad (\text{A.2})$$

The solution to this differential equation can also be written as the convergent sum

$$M_{k,m}(z) = z^{m+\frac{1}{2}} e^{-\frac{z}{2}} \sum_{n=0}^{\infty} \frac{(m-k+1/2)_n}{n!(2m+1)_n} z^n$$

where $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$ is once again the Pochhammer symbol, given in [Abramowitz and Stegun, 1972, p. 256].

A.4 The One-Dimensional Dunkl Kernel

The Dunkl kernel is used in the thesis predominantly to express the semigroup density of the Dunkl process. This is a process whose generator arises as a result of using the perturbed Dunkl operator as opposed to the usual differential operator in the Laplacian operator from which we obtain Brownian motion. In the one-dimensional case, an explicit expression for the Dunkl kernel can be found, see Gallardo and Yor [2006],

$$D_\nu(x) = \frac{1}{B(\frac{1}{2}, \nu)} \int_{-1}^1 e^{ux} (1-u)^{\nu-1} (1+u)^\nu du$$

where B is the Beta function.

In addition, the one-dimensional Dunkl kernel can also be expressed in terms of the modified Bessel function:

$$D^{(\nu, \lambda)} = \begin{cases} \frac{z^{-\nu}}{2} \left(I_\nu(z) + I_{\sqrt{\nu^2+4\lambda}}(z) \right) & \text{if } z \geq 0 \\ \frac{|z|^{-\nu}}{2} \left(I_\nu(-z) - I_{\sqrt{\nu^2+4\lambda}}(-z) \right) & \text{if } z < 0. \end{cases} \quad (\text{A.3})$$

This expression arises from using the representation of the Dunkl process as a process that switches between a positive and a negative Bessel processes at times decided by a time-changed Poisson process and is detailed in Chybiryakov et al. [2008].

Appendix B

The Spherical Coordinates Notation

The self-similar property and many of the restrictions on the semigroup density in Lawi [2008] are with respect to a scalar variable ($\lambda > 0$) and are therefore challenging to apply to Cartesian coordinates, which do not generally satisfy the scalar properties in higher dimensions. For this reason, in Chapters 3 and 4, we would like to move to spherical coordinates so we recall this change of variables here.

Moving to the spherical coordinate system involves a bijective change of variable where every point is determined through its angular and radial parts rather than its distance from the axes. For example, for a $y = (y_1, y_2)^T \in \mathbb{R}^2$ and a $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$ in Cartesian coordinates the spherical coordinates can be expressed

$$\begin{aligned} y_1 &= r \cos(\phi^{(y)}) & r &= \sqrt{y_1^2 + y_2^2} \\ y_2 &= r \sin(\phi^{(y)}) & \phi^{(y)} &= \arctan\left(\frac{y_2}{y_1}\right) \end{aligned}$$

for $r \in [0, \infty)$, $\phi \in [0, 2\pi)$ and

$$\begin{aligned} z_1 &= r \cos(\phi_1^{(z)}) & r &= \sqrt{z_1^2 + z_2^2 + z_3^2} \\ z_2 &= r \sin(\phi_2^{(z)}) \sin(\phi_1^{(z)}) & \phi_1^{(y)} &= \arccos\left(\frac{z_1}{\sqrt{z_1^2 + z_2^2 + z_3^2}}\right) \\ z_3 &= r \cos(\phi_2^{(z)}) \sin(\phi_1^{(z)}) & \phi_2^{(y)} &= \arctan\left(\frac{z_2}{z_1}\right) \end{aligned}$$

for $r \in [0, \infty)$, $\phi_2 \in [0, 2\pi)$ and $\phi_1 \in [0, \pi)$.

Furthermore, we would also like to recall from the work of Blumenson [1960] that, in the general case of spherical coordinates in n -dimensions for a particular point $y = (y_1, \dots, y_n)^T$ in \mathbb{R}^n , we decompose it into its radial part r and its angular parts $\underline{\phi}_y = (\phi_1^{(y)}, \dots, \phi_{n-1}^{(y)})^T$ in the

following way:

$$\begin{aligned}
y_1 &= r \cos(\phi_1^{(y)}) \\
y_j &= r \cos(\phi_j^{(y)}) \prod_{k=1}^{j-1} \sin(\phi_k^{(y)}) \quad \text{for } j = 2, \dots, n-2 \\
y_{n-1} &= r \cos(\phi_{n-1}^{(y)}) \prod_{k=1}^{n-2} \sin(\phi_k^{(y)}) \\
y_n &= r \sin(\phi_{n-1}^{(y)}) \prod_{k=1}^{n-2} \sin(\phi_k^{(y)}),
\end{aligned}$$

where $r \in [0, \infty)$, $\phi_{n-1} \in [0, 2\pi)$ and $\phi_i \in [0, \pi)$ for all $1 \leq i \leq n-2$. For the ease of notation we refer to this as $y = rg(\underline{\phi})$ where $g : [0, 2\pi] \times [0, \pi]^{n-1} \rightarrow S^{n-1}$, where S^{n-1} is the $(n-1)$ -dimensional sphere on \mathbb{R}^n . Moreover, the bijective nature of this construction allows us to make the integral substitution

$$dy_1 \dots dy_n = r^{n-1} \prod_{k=1}^{n-2} \sin^k(\phi_{n-1-k}^{(y)}) dr d\underline{\phi}_y,$$

which we refer to as $dy_1 \dots dy_n = r^{n-1} h(\underline{\phi}_y) dr d\underline{\phi}_y$.

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